NATIONAL ACADEMY OF SCIENCES

JOHN VON NEUMANN

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A Biographical Memoir by

S. BOCHNER

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JOHN VON NEUMANN's academic career was in mathematics, even though he was very much associated with theoretical physics and to an extent with theoretical economics. He began his career in earnest with axiomatic set theory and logic and pursued this topic actively for several years. In the closing years of his life, while busily filling many positions as scientific executive on a national level, he was laying plans for a treatise in the "logics" underlying the mathematical foundations of games and strategies with applications to economics, of the theory of codifying instructions to computing machines, and of the theory of networks both electrical and nervous. The activities mentioned did not complete his professional life by any means. But in a sense they were very characteristic of the man and his basic style.

The contemporary mathematical activities we have just enumerated, in which von Neumann so excelled, whatever their implications for mid-century's modern man and whatever their fascination for the philosopher in search of modernity, are not yet too well established with the old-line mathematician, and the density of their mathematical content is as yet not always deemed to be optimal. In fact, von Neumann's mathematician's mathematics was of stronger texture by its very nature, and his prowess in theoretical physics also had to maintain itself rather more competitively. But his lifelong commitment to questions of logics, axiomatization, and formalization was expressive of his mathematical personality, nonetheless.

In his more traditional achievements he had his peers, and in his work in theoretical physics he did not originate a basic verity as several of his contemporaries in the 1920s did. But if to his preponderance in mathematics and physics, and his familiarity with many other problems of science, one adds his momentous excursion into the theory of rational behavior—as the implications of games and strategies are named by some—then it can be said that, among mathematicians at any rate, he was an exceptional contemporary indeed. By intellectual faculty he was a mathematician, but by species he was a scientific man within all the major connotations of today, from esoteric ones to those of popular appeal.

His effectiveness was largely due to his ever-present mental manipulatory quickness. He could literally "think on his feet," and much of his best work may have received its initial impulse in just this way. He had a prodigious memory, and legend has it that he knew all the facts and dates from many volumes of standard histories by heart.

Apart from achievements in the areas enumerated above, the high points of his work are as follows: (1) the extension of operators in Hilbert space from bounded to unbounded ones (# 30 of the Bibliography), (2) almost periodic functions on groups (# 52), and (3) his masterwork, algebras of bounded operators in Hilbert space (beginning with # 62). To these one might add (4) his definition of a "state" in quantum theory as an element in a Hilbert space (beginning with # 9) and (5) the introduction of analytic parameters in a locally Euclidean group in the compact case (# 48).

1. His theory of non-bounded operators in Hilbert space was arresting. The naturalness with which he axiomatized the Hilbert space and introduced concepts like maximal operator, hypermaximal operator, deficiency of an operator, etc., was breathtaking to specialist and nonspecialist alike. And yet there was nothing greatly contrived in the analysis, nor was there anything that some of the other mathematicians who were familiar with the topic could not have really done. As a matter of fact, Frederic Riesz, who had known the theory

as well as anybody, after catching his breath wrote a comment on the new theory, in the introduction to which he asks himself somewhat plaintively what he, Frederic Riesz, had been doing with himself all these years.

2. His theory of almost periodic functions on groups was glamorous. It soon turned out that syllogistically the theory can be put together much more simply than von Neumann had done it. But the effect of the theory, especially on the so-called Abstract Harmonic Analysis which was soon to emerge, was a decisive one, and he fully deserved obtaining the Bôcher prize for it.

3. The theory of algebras of operators, written in important phases in collaboration with F. J. Murray, has not the ease and naturalness of much of his earlier work and it is somewhat heavy in the manner of composition, but in strength, tenacity, and importance it is not outdone by anything in its generation. Any representation of a group by bounded operators (in an infinitely dimensional space) can be enlarged to such a representation of the enveloping group algebra and thus subsumes under the Murray-von Neumann theory. Now it is a fact that the theory of representations in the particular case of a group algebra has been and continues to be strongly guided by the more general theory, and in certain parts must still rely on taking over results directly from it.

4. Dr. von Neumann's papers in the 1920s on the foundations of quantum mechanics and their effect on thermodynamics were extremely mature. His formal definition of a physical "state" as being a point in a Hilbert space representing the system was as readily accepted by physicists as if it had been the most obvious thing since creation, and this concept of a "state" has had some repercussions in a purely mathematical context as well.

5. His solution of "Hilbert's fifth problem" in the compact case had no bearing on the subsequent solution by others in the non-compact case, but it is nonetheless interesting as an achievement of von Neumann's. He shows in it his mastery of finite linear representations, and he employs in it the group invariant measure which had

just been found by Haar. It is somewhat surprising that von Neumann did not construct the group invariant measure himself, since measure theory, as a bridge between Boolean algebras and linear functionals on vector spaces, was very close to his thinking. But at any rate, soon after this measure had been introduced, he studied it extensively and contributed much to its propagation.

Dr. von Neumann also attached major standing to his work on the ergodic theorem, 1931–32 (#41). On one occasion, he listed his essential work under the following three headings: the papers on quantum mechanics in the 1920s; his total work in operator theory, 1930–39; his version of the ergodic theorem.

Most of von Neumann's published work involves in a decisive phase linear spaces, finite or infinitely dimensional ones, or functionals and linear operations between such. The approach via linear spaces covers, or can be made to cover, a large part of the total mathematical area nowadays. Still, an important part of the landscape is not included in it. It leaves out almost anything which involves "geometry" in a proper global sense, such as topology, differential geometry and harmonic integrals, general relativity globally, algebraic geometry, etc. Into this kind of mathematics von Neumann ventured rarely, if at all, and he actually had no affinity with it in his thinking. An incident in his work may illustrate this. In 1929 he published the important result (# 24) that a group of linear transformations in a finite dimensional space, if closed, is a Lie group. Élie Cartan, who was a geometer, immediately recognized that the theorem is a special case of the general proposition that a closed subgroup of a Lie group is again a Lie group, and he proved it by the same procedure, although in an abbreviated form. There is hardly a young mathematician today who realizes that this familiar general proposition was von Neumann's discovery originally.

The various parts of mathematics which involved geometry have of late been tending to coalesce into one, and thus to restratify the classical traditional body of mathematics into a new pattern. This is a significant development, and to a historical observer it would almost appear as if this mathematics were harking back to its Pythagorean origin, when all mathematics was geometry and "structure" in dissoluble union, by deliberate design, and less naively so than is sometimes assumed. On the other hand, the mathematics of logical and operational approach, to which von Neumann was gravitating by natural disposition, is also aware of its origin genetically, but in a rather more negating manner on the whole, and in a much more consciously modern and even antagonistic spirit. Many a chapter in mathematical logic or semantics is like a page from Aristotle's *Organon* or *Rhetoric,* but with all sentences crossed out and each replaced by an entirely different one on the margin.

In short, there are two trends of modernism discernible in mathematics today. One is incisive, emphatic, frequently stirring, and always provocative. The other is a kind of regeneration with an element of traditionalism operating as a positive motive power in it. Von Neumann did not share in the latter.

John von Neumann always had the natural air of someone who had been reared from childhood in material comfort. He was born on December 28, 1903, in Budapest, Hungary, the eldest of three sons, and first came to the United States when he was twenty-seven years old. His brothers and his widowed mother also came here, and his mother died during von Neumann's illness, only six months before his death. His father, Max von Neumann, was a banker. First manager and later vice-president of a major bank, he finally was made one of the partners of a private banking firm, all in Budapest. According to John, it was his father who encouraged him to choose a career in science. John showed a pronounced interest in mathematics from the age of thirteen. He was also a great reader of books on history throughout his life, and in both science and history his retentive memory was most remarkable.

He attended grammar school and high school in Budapest. As a youth he had no bent towards physical or sporting activities. And even when later on he drove some very dashing looking cars with the maximum velocity for which they were designed, he never acquired the mien of a "sporting type." He did claim, however, that he had once learned to ride a bicycle, but admitted that a pair of trousers was part of the "tuition price." In his teens he was somewhat gauche and not quite the type of "leader." But there was nothing awkward about his manner in later life, and he gave the impression of considerable collectedness and self-confidence. He was courteous and could be quite jovial. In company he was an excellent raconteur. He never smoked, and would drink but rarely, although he would sometime simulate gaiety as if he had imbibed.

Even before entering on college studies, he was co-author of a paper (# 1 of the biblography) at the age of eighteen. He studied two years at the University of Berlin, 1921–23, and also took courses in chemistry. Fritz Haber, the then academic leader of chemistry in Germany, soon met him, and is reported to have said to friends that he expected or wished that von Neumann would take up chemistry as a career. John went from Berlin to the Institute of Technology in Zürich, Switzerland, where Hermann Weyl was in the mathematics department at the time. He left the Institute in 1925 with an engineering degree in chemistry. In the meantime, he had begun publishing very good mathematical papers pertaining to axiomatic set theory, and in 1926 he was awarded a doctorate for such a thesis at the University of Budapest. This enabled him to secure immediately a position as Privatdozent at the University of Berlin. One of the ranking mathematicians there was Erhard Schmidt, known mainly for his work in integral equations, who also had a strong interest in set theory, fostered by a lifelong friendship with Ernst Zermelo, of "Zermelo-axiom" fame.

John's interest soon turned to the recently created quantum theory, which was produced, packaged, and delivered by several "youngsters" in their twenties, barely older than himself. He began to think through the mathematical scheme of the theory in his own terms and, in the process, learned theoretical physics, and readied mathematical tools for future use. He wrote some lengthy articles on quantum theory, and there are indications in the first ones that it was Max Born's statistical interpretation of the Schroedinger wave function and also Born's attempt at rigorization of approach that put von Neumann initially on the track of becoming one of the busiest and most sought-after scientific consultants to military and other organizations in this country.

In 1928, between papers on mathematical logic and his first papers in physics, he "squeezed in" his first and basic paper on the theory of games. There had been forerunners in the field (E. Zermelo, H. Borel, and H. Steinhaus, among others) by whom most of the concepts like play and counterplay, chance and deception, pure strategy and mixed strategy had been anticipated in one form or another. But von Neumann brought the results of others and his own new ones together, clearly and axiomatically, into twenty-five pages. In the briefest of footnotes he made reference to the analogy with some problems in economics, and in another footnote, added in proof, he made reference to a very recent note of Borel's in which for the two-person zero-sum game the underlying minimax theorem was proposed, although not proven, as in von Neumann's paper. Eventually the paper grew into the treatise with Oscar Morgenstern published by the Princeton University Press in 1944 (# 90). When deciding to print the book, the Press was cheerfully resigned, for the sake of scholarship, to face the financial deficit they expected. But, in fact, they had to reprint the book in 1947 and again in 1953.

The quantum theory led von Neumann naturally to a re-examination of the theory of operators in Hilbert space and, while still in Berlin, he published in 1929 his paper # 30 on unbounded operators. It was soon followed by his paper # 31 on algebras of operators, the first in that direction. He discussed this material with Erhard Schmidt, who had some influence on clarifying the distinction between maximal (symmetric) and hypermaximal (selfadjoint) operators. This last work won for him the admiration of mathematicians. In 1929 he transferred to the University of Hamburg as Privatdozent, but remained only one year. In 1930 he came to Princeton, N. J., which remained his academic address afterwards.

About that time the Springer publishing firm solicited him to write for their so-called Yellow Collection (this was not the official name) a volume on the mathematical foundations of quantum mechanics (# 47a). While negotiations for the book were still in progress, von Neumann became dissatisfied with the galley proofs of an article in print, and wanted to make very costly revisions in it. The journal in which the article was to be published was also controlled by Springer, and in order to humor him they let him make the revisions without charge.

Just before coming to Princeton in 1930, von Neumann married Marietta Kovesi in Budapest. They were divorced in 1937, shortly after their daughter Marina was born. She is his only child, and has been Mrs. Robert Whitman since June 1956. In 1938, von Neumann married Klara Dan, also originally from Budapest, who later worked on his Computer Project.

From 1930 until 1933 he was Visiting Professor at Princeton University, and in 1933 became a permanent member of the Institute for Advanced Study, which academic post he held till his death. He soon became widely known to mathematicians in this country, at first chiefly to senior ones, but after his paper on almost periodic functions (# 55) to the "rank-and-file" as well. Until 1940, when the war began to change the pattern of his activities, he gave lecture courses fairly regularly, and he had large audiences. Furthermore, his papers were written in a uniform orderly style and were accessible for study by younger research workers, even though the subject was a difficult one.

In 1935, when F. J. Murray was a National Research Fellow in residence at the Institute for Advanced Study, von Neumann began the study of algebras of operators in earnest. Several of the papers were written in collaboration (# 62, # 70, # 87) with Murray, and Murray's partnership was not a nominal one. There were many discussions between them with a brisk give-and-take throughout. An offshoot of these studies was von Neumann's "continuous geometries." They are typified by a kind of projective space in which the hierarchy of linear subspaces of increasing dimension is no longer specified by discrete dimension numbers (that is, integer numbers, or multiples of integer numbers with some common factor) but by continuous real numbers out of an entire interval. This was the topic of his Colloquium Lectures before the American Mathematical Society in 1937. He did not publish the lectures, as most lecturers do, but several sets of notes found among his papers, written as always in his legible round hand, may have been intended to be used for this purpose.

Dr. von Neumann was elected to the National Academy of Sciences in 1937.

After completing the book with Morgenstern in 1944, von Neumann's attention turned to computing machines and, somewhat surprisingly, he decided to build his own. As the years progressed, he appeared to thrive on the multitudinousness of his tasks. It has been stated that von Neumann's electronic computer hastened the hydrogen bomb explosion of November 1, 1952.

When the war came, von Neumann was called to serve on an increasing number of upper echelon boards and committees concerned with armament programs or the utilization of nuclear energy. As most of the research work involved was classified, it is not easy to assay what kind of original work he did. But all published citations which he received are unanimous and convincing in praising him for outstanding contributions in many areas of vital importance.

In the summer of 1955, only a few months after his appointment to the Atomic Energy Commission had been approved by the Senate, he became ill. Cancer was soon suspected, but he continued to work fairly regularly for several more months. His last public appearance was early in 1956 when, in a wheelchair at the White House, he received the Medal of Freedom from President Eisenhower. About that time it began to be known that he was not expected to recover. In April, 1956, he was taken to the Walter Reed Hospital, where he remained till the end. He lingered on longer than was expected, dying on February 8, 1957.

Von Neumann was baptized a Roman Catholic in 1930, and he died in the faith. He is buried in Princeton, N. J.

CHRONOLOGY

1903 Born, Budapest, Hungary, December 28.
1930–33 Visiting Professor, Princeton University.
1933–57 Professor of Mathematics, Institute for Advanced Study, Princeton, N. J.
1937 Gibbs Lecturer, Colloquium Lecturer, Bôcher Prize, all in American Mathematical Society.
1940–57 Scientific Advisory Committee, Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland.
1941–55 Navy Bureau of Ordnance, Washington, D. C.
1943–55 Los Alamos Scientific Laboratory (AEC), Los Alamos, N. M.
1945–57 Director of Electronic Computer Project, Institute for Advanced Study, Princeton, N. J.
1947 D. Sc. (hon.), Princeton University; Medal for Merit (Presidential Award); Distinguished Civilian Service Award, U. S. Navy.
1947–55 Naval Ordnance Laboratory, Silver Spring, Maryland.
1949–53 Research and Development Board, Washington, D. C.
1949–54 Oak Ridge National Laboratory, Oak Ridge, Tennessee.
1950 D. Sc. (hon.), University of Pennsylvania and Harvard University.
1950–55 Armed Forces Special Weapons Project, Washington, D. C.; Weapons System Evaluation Group, Washington, D. C.
1950–57 Member Board of Advisors, Universidad de los Andes, Colombia, South America.
1951–53 President, American Mathematical Society.
1951–57 Scientific Advisory Board, U. S. Air Force, Washington, D. C.
1952 D. Sc. (hon.), University of Instanbul, Case Institute of Technology, and University of Maryland.
1952–54 Member, General Advisory Committee, U. S. Atomic Energy Commission, Washington, D. C. (Presidential appointment).
1953 D. Sc. (hon.), Institute of Polytechnics, Munich; Vanuxem Lecturer, Princeton University.
1953–57 Technical Advisory Panel on Atomic Energy, Washington, D. C.
1955–57 U. S. Atomic Energy Commissioner (Presidential appointment).
1956 Medal of Freedom (Presidential award); Albert Einstein Commemorative Award; Enrico Fermi Award.
1957 Died, Washington, D. C., February 8.
Academy memberships:
Academia Nacional de Ciencias Exactas, Lima, Peru.
Academia Nazionale dei Lincei, Rome, Italy.
American Academy of Arts and Sciences.
American Philosophical Society.
Instituto Lombardo di Scienze e Lettere, Milan, Italy.
National Academy of Sciences.
Royal Netherlands Academy of Sciences and Letters, Amsterdam, Netherlands.

KEY TO ABBREVIATIONS

Acta Szeged. = Szeged. Egyetem. Acta universitatis szegediensis. Acta scientarum mathematicarum.
A. M. S. Bull. = American Mathematical Society Bulletin
A. M. S. Trans. = American Mathematical Society Transactions
Ann. Math. = Annals of Mathematics
Ann. Math. Stat. = Annals of Mathematical Statistics
Ann. Math. Studies = Annals of Mathematics Studies
Astrophys. J. = Astrophysical Journal
Commun. Pure Appl. Math. = Communications on Pure and Applied Mathematics
Compos. Math. = Compositio Mathematica
Erg. eines Math. Coll. = Ergebnisse eines mathematischen Kolloquiums
Fund. Math. = Fundamenta Mathematicae
Gött. Nach. = Gesellschaft der Wissenschaften zu Göttingen Nachrichten. Mathematisch-physikalische Klasse.
Hamb. Abh. = Hamburg Abhandlungen
Jahresb. = Deutsche Mathematiker-Vereinigung, Leipzig, Jahresbericht
J. Appl. Phys. = Journal of Applied Physics
J. f. Math = Journal für Mathematik
Math. Ann. = Mathematische Annalen
Math. Nach. = Mathematische Nachrichten
Math. Tables and Other Aids to Comp. = Mathematical Tables and Other Aids to Computation
Math. Zschr. = Mathematische Zeitschrift
Math. Phys. Lapok = Mathematikaies physikai Lapok
Math. Sbornik = Matica slovenska, Turócz-Szent-Márton. Sbornik.
N. A. S. Proc. = National Academy of Sciences Proceedings
Nat. Bur. Standards Appl. Math. Series = National Bureau of Standards Applied Mathematics Series
Phys. Zschr. = Physikalische Zeitschrift
Portugaliae Math. = Portugaliae Mathematica
Proc. Comp. Sem. = Proceedings, Comptroller's Seminar
Rev. Econ. Studies = Review of Economic Studies
Sitzungsber. d. Preuss. Akad. = Sitzungsberichte der Preussischen Akademie
Tomsk. Univ. Rev. = Tomsk Universitet Review
Zschr. f. Phys. = Zeitschrift für Physik

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1923

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1925

3. Eine Axiomatisierung der Mengenlehre. Jour. f. Math., 154:219–40.
4. Egyenletesen sürü számsorozatok. Math. Phys. Lapok, 32:32–40.

1926

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1927

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1928

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