



Norman Levinson

1912–1975

BIOGRAPHICAL

Memoirs

*A Biographical Memoir by
D. G. Aronson*

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NATIONAL ACADEMY OF SCIENCES

NORMAN LEVINSON

August 11, 1912–October 10, 1975

Elected to the NAS, 1967

Norman Levinson was a preeminent American mathematician who made fundamental contributions to complex function theory, differential equations, and analytic number theory. He was born in Lynn, Massachusetts, to a family of poor Russian-Jewish immigrants. His father was a shoe factory worker who earned three dollars a week and whose education consisted of having attended a yeshiva for a few years. His mother was illiterate. When his father changed jobs, the family moved to the nearby town of Revere; there they bought a small house that had no bathroom and was heated by an oil stove in kitchen. But despite poverty and a low educational level, Norman's parents managed to send him and his sister Pauline to college. "We were very poor," he recalled, "but we didn't think of ourselves as poor."



By D. G. Aronson

Photography courtesy MIT Museum.

Levinson attended Revere High School, where, having been diagnosed with rheumatic fever, he was unable to participate in any athletic activity. This illness probably contributed, moreover, to his lifelong hypochondria. To help the family finances, he worked evenings in a neighborhood grocery store.

In 1929 Levinson entered the Massachusetts Institute of Technology and majored in electrical engineering. He received his B.S. and M.S. degrees in electrical engineering in 1934. While still an undergraduate he took almost all the graduate mathematics courses offered at MIT and wrote a thesis under Norbert Wiener. According to H. D. Phillips, the head of the mathematics department, this document had "results sufficient for a doctoral thesis of unusual excellence."

One of the mathematics courses that Levinson took in the 1933–34 academic year was Wiener's course on Fourier series and integrals, which proved to be a turning point in Levinson's career. As he described it:

*I became acquainted with Wiener in September 1933, while still a student of electrical engineering, when I enrolled in his graduate course. At that level he was a most stimulating teacher. He would actually carry on his research at the blackboard. As soon as I displayed a slight comprehension of what he was doing, he handed me the manuscript of Paley-Wiener [a book, *Fourier Transforms in the Complex Plane*, that Wiener coauthored with Raymond E. A. C. Paley] for revision. I found a gap in a proof and proved a lemma to set it right. Wiener thereupon sat down at his typewriter and sent it off to a journal. A prominent professor does not often act as a secretary for a young student. He convinced me to change my course from electrical engineering to mathematics. He then went to visit my parents—unschooled immigrant working people living in a run-down ghetto community—to assure them about my future in mathematics.[1]*

After completing his B.S. and M.S. degrees in electrical engineering, Levinson applied to become an MIT doctoral student in mathematics. But because the mathematics department felt that he had already done sufficient work for a doctorate, Wiener and Phillips arranged for him to receive an MIT Redfield Proctor Traveling Fellowship and assured him that he would receive his doctorate at the end of his traveling year. Levinson used the fellowship to spend the year in Cambridge, England, to work with G. H. Hardy. His parents were not happy with his plan to go overseas, but Wiener visited them again and convinced them of the wisdom of the move. Levinson's year in Cambridge was very productive, although there is some question about how much contact he actually had with Hardy. Nevertheless, Hardy was certainly familiar with Levinson's work, as much of it involved extensions of and improvements on some of Hardy's results.

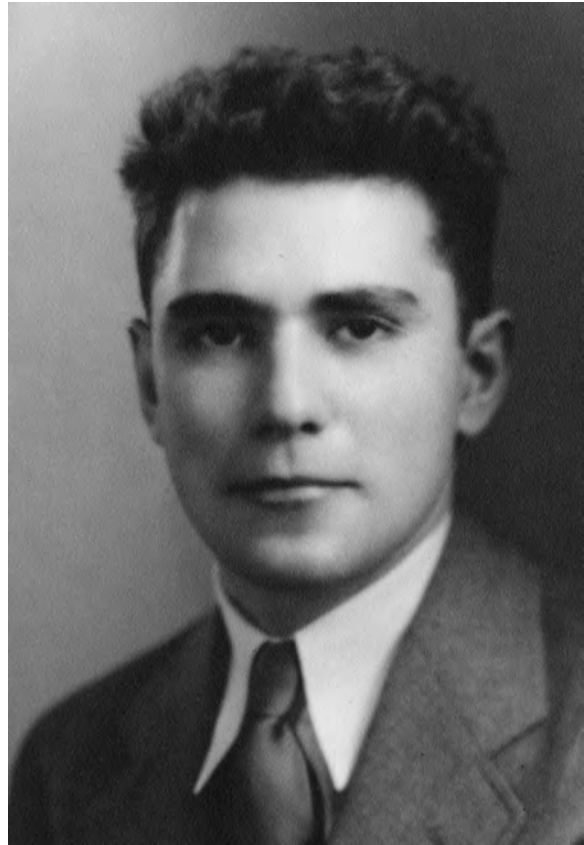
According to Levinson's wife Fagi:

Vannevar Bush took [Hardy] around and every time he showed him a new thing, Hardy said, "What a marvelous theological institution this is!" Finally, [Bush] said, "It's not a theological institution!" And Hardy said, "Then why don't you hire Levinson?"

Upon his return to MIT in 1935, Levinson received his Ph.D. degree for a thesis titled *Non-Vanishing of a Function*. He was then awarded a National Research Council Fellowship for a two-year stay at the Institute for Advanced Study (IAS) in Princeton, New Jersey, with John von Neumann as his nominal supervisor.

Because of widespread unemployment and anti-Semitism in the United States during the mid-1930s, the job prospects for Jewish mathematicians were very bad. The situation was further complicated by the influx of high-quality Jewish mathematicians fleeing Germany and its neighbors from 1937 on. However, in 1936 Jesse Douglas (a Fields Medal winner) became ill and was unable to teach his courses at MIT. Wiener recommended that Levinson be hired to replace Douglas, but anti-Semitism at MIT thwarted this effort; the school's administration turned down Wiener's recommendation. Hardy, who had come to Princeton for its 200th anniversary celebration, also visited MIT. There are several versions of the following encounter. According to Levinson's wife Fagi:

Vannevar Bush took [Hardy] around and every time he showed him a new thing, Hardy said, "What a marvelous theological institution this is!" Finally, [Bush] said, "It's not a theological institution!" And Hardy said, "Then why don't you hire Levinson?"[2]



Levinson in 1938.

(Photo courtesy MIT Museum.)

Levinson was appointed as an instructor at MIT in February 1937, leaving his IAS fellowship before its term was complete, and was promoted to assistant professor in 1939. Meanwhile, he married Zipporah Wallman (generally known as Fagi) on February 11, 1938. She was the sister of the topologist Henry Wallman. They had two daughters, Sylvia in 1939 and Joan (Zorza) in 1941.

From 1935 until about 1940 Levinson devoted himself to research in harmonic and complex analysis. His work was greatly influenced by the methods of Paley and Wiener, as expounded in *Fourier Transforms in the Complex Plane*, particularly in his use both of real and complex variable techniques. This was a tremendously productive period for Levinson, resulting in more than 15 publications. Many of these results, together with numerous others, were collected in his 1940 book *Gap and Density Theorems*, published by the American Mathematical Society in its Colloquium Publication Series (Volume 29). Topics covered in the book included the completeness of sequences of complex exponentials, the study of Fourier transforms that vanish over an interval, analytic functions whose rate of growth along a line is determined by their growth on a sequence of points on that line, and some properties of Dirichlet series. The book concluded with a striking generalization of a Tauberian theorem due to Hardy and John E. Littlewood. In his review in *Mathematical Reviews*, G. Polya stated, “The author contributes something essential to all his subjects, obtains very precise results, and gives new proofs. Some of his proofs are long and difficult, but the details are presented with much care and precision.” An extremely detailed summary of the book’s content and the subsequent work it inspired can be found in Raymond Redheffer’s essay in Volume 2 (1997) of the *Selected Papers of Norman Levinson* (Boston-Basel-Berlin: Birkhäuser).

Levinson’s style could be characterized as old-school hard analysis. Particularly in his early papers, his proofs tended to be long and computationally intensive. Moreover, his writing was quite austere. There was often little or no introductory material. Rather he would go almost directly to the statement of his results and their proofs. Nevertheless, his work was very insightful and amply rewards the effort involved in following his arguments. As an aside, I note that when I was preparing to write up my dissertation under Levinson’s supervision in 1955, he told me to omit all introductory material and give only the results and proofs. Consequently, I had to pad my thesis for publication!

With the exceptions of his year at Cambridge, his IAS fellowship, and his year as a Guggenheim fellow in Copenhagen (1948–1949), Levinson spent his whole academic career at MIT. Starting as an undergraduate in 1929 he became an instructor (1937–1939), an assistant professor (1939–1944), an associate professor (1944–1949), professor (1949–1971), and Institute Professor (1971–1975).

Living at the height of the Great Depression, Levinson was highly disturbed by America’s rampant unemployment, anti-Semitism, and discrimination against blacks. The American Communist Party shared these sentiments, and Levinson joined in 1937.

He gradually became disenchanted with communism, however—particularly with the excesses of the Soviet regime—and by 1945 had drifted out of the party. But he retained a lifelong concern for social justice. In 1953, Levinson and two other MIT faculty were called to testify before the House Un-American Activities Committee. Much to its credit, MIT provided the three professors with highly skilled legal counsel. The committee wanted the three to name fellow party members, but while the other two were willing to do so, Levinson refused to comply. The lawyer arranged with the committee to have the other two testify first and require Levinson to testify only about his own activities. The following was his final statement to the committee in response to a question:

Mr. Kunzig [committee council]: "...Professor, you said very emphatically that the danger today was the hard core of underground members of the Communist Party in America. Isn't it true then, that every Communist who is exposed, who is brought to the light of the American people, is lessening that danger? In other words, that helps America?"

Dr. Levinson: "Yes, I would agree, with one very important qualification—that it's very important to make sure that it's a real Communist that is being exposed. The naming of men, innocent men, or men who have long left the Communist Party, or who have never been Communist Party members—I think that is a terrible thing. I think that any harm to nonguilty [sic] people, no matter how desirable it is to catch guilty people, should be avoided. I think—I think it is the American way—not to hurt the innocent. I think every effort should be made, in exposing the guilty, to avoid doing any damage whatsoever to people who were never communists, or there may be speculation about them but no real evidence, or people who have left the Communist party, and so on. I think this is a very important thing." [3]

Because of the attendant publicity, the years immediately following their testimony were very difficult for the three professors. However, MIT stood behind them and took no punitive action.

After the publication of his book *Gap and Density Theorems*, Levinson continued to work on and off in complex analysis; and in 1970, in collaboration with his former student R. M. Redheffer, he published a textbook titled *Complex Variables*. However, he also decided to switch his main field to the study of linear and nonlinear ordinary and partial differ-

ential equations. This field was essentially new to him and so it required considerable effort to get up to speed.

Levinson's work on differential equations was many-faceted, and he made numerous fundamental contributions to the field. In 1953 he received the Bôcher Memorial Prize from the American Mathematical Society. The citation reads: "Eighth award to Norman Levinson for his contributions to the theory of linear, nonlinear, ordinary, and partial differential equations contained in his papers of recent years."

In describing Levinson's work on ordinary differential equations, we begin with his research on stability and asymptotic behavior of solutions of linear systems. Here there were two main problems. The first concerned the relationship between the solutions of the system

$$x' = Ax \tag{1}$$

where A is a constant $n \times n$ matrix, x and $x' = dx/dt$ are n -vectors; and the variable coefficient system

$$y' = F(t)y. \tag{2}$$

The question is: When does the solution of the initial value problem for (2) approach a solution of (1) as t tends to infinity? Clearly it is necessary that $F(t)$ approximate A for large values of t . In [4] and [5] Levinson derived very precise conditions on $F(t)$ that guarantee the desired behavior. His results generalized and subsumed all previous results on this class of problems. Moreover, his results and methods had a significant impact on further progress. This was made abundantly clear in M. S. P. Eastham's 1989 monograph *The Asymptotic Solution of Linear Differential Systems: Applications of Levinson's Theorem*, which summarized the research that had been done since Levinson's original publications in the 1940s. More recent research [6] established the connection with the so-called Evans function, which plays a central role in the study of stability of certain traveling-wave solutions of general parabolic systems.

The second problem concerned, the nonlinear system

$$z' = Az + f(z, z', t),$$

where A is a $n \times n$ constant matrix and z , z' , and f are n -vectors. It is assumed that all the eigenvalues of A have negative real parts and that, for example, $f(z, m, t) = o(|z| + |m|)$

uniformly for t nonnegative. Then if $|z(0)|$ and $|z'(0)|$ are sufficiently small, $|z(t)|$ and $|z'(t)|$ are uniformly bounded for nonnegative t and tend to zero as t tends to infinity. [7] In particular, this result generalizes a theorem of R. Bellman [8] as well as results of other researchers.

In 1927, B. Van der Pol [9] studied the nonlinear differential equation

$$\ddot{x} - m(1 - x^2)\dot{x} + x = 0,$$

where m is a parameter. This equation arose from problems in electrical engineering, and the question of interest is the existence and uniqueness of periodic solutions. The term $m(1 - x^2)$ represents damping and it is clear that, with $m > 0$, the damping is negative for $|x| < 1$ and positive for $|x| > 1$. Using graphical methods, Van der Pol showed that, for various values of m , solutions exist and rapidly approach a steady-state oscillation as t tends to infinity. A. Liénard [10] generalized this result to the case in which the damping term is replaced by $f(x)x$ with suitable restrictions on $f(x)$.

Levinson and his former student Oliver K. Smith [11] considered the more general equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0.$$

Assuming that $xg(x)$ is nonnegative, and that $f(x, v)$ is positive for large $|x|$ and negative for small $|x|$ and $|v|$, they established the existence of periodic solutions. The question of uniqueness requires further restrictions. For example, they proved uniqueness for the particular case

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x)$ is not necessarily even and $g(x)$ is not necessarily odd. Extensions of this work to equations with periodic forcing are due to Levinson and various other authors.

In 1945, M. L. Cartwright and J. E. Littlewood [12] published a paper on a version of the forced Van der Pol equation that arose from the study of oscillations in a nonlinear network. They indicated the occurrence of several unexpected phenomena, including the existence of discontinuous recurrent motions. However, their proofs were quite sketchy. This paper triggered Levinson's interest in the subject. The equation he considered was essentially the same as the Cartwright-Littlewood equation except for the replacement of the nonlinear term by a piecewise linear term [13]. Specifically, the equation was

$$a\ddot{x} + p(x)\dot{x} + ax = b \sin(t),$$

where a is a small constant, $p(x) = 1$ for $|x| > 1$ and $p(x) = -1$ for $|x| < 1$, and the constant b must be chosen from a certain set of subintervals of the unit interval. This equation exhibits the same range of phenomena as the Cartwright-Littlewood equation, and the introduction of piecewise linearity greatly clarifies the exposition. Levinson proved that the Poincaré map associated with this equation possesses a singular attractor. It is a set of measure zero and is not a Jordan curve. Moreover, it contains infinitely many periodic orbits as well as discontinuous recurrent motions. Jürgen Moser stated:

One must stress that Levinson's paper was not just a verification of a previously known result. The work of Cartwright and Littlewood in [(12)] and their subsequent papers was sketchy...[and] quite obscure, as Littlewood freely admitted. When I asked [Littlewood] about it in June 1977, he said, "Ah. you mean the monster paper [(12)]; it had been read by only three people in the world: the authors and [Peter] Swinnerton-Dyer." So it was a major achievement for Levinson to provide a definitive proof of this result [14].

In 1959, Steven Smale conjectured [15] that a structurally stable dynamical system in three or more dimensions could not have infinitely many periodic orbits. To Smale's surprise [16], Levinson's work provided an example to show that this conjecture was false. Smale analyzed Levinson's paper and was led to the concept of the horseshoe map, which has had a great impact on the development of the modern geometric theory of dynamical systems. In 1981 Mark Levi [17] took up the study of Levinson's equation. He greatly clarified the results of Levinson and Smale, and he solved as well some problems left open by Littlewood. Levi proved that Levinson's equation possesses infinitely many stable periodic solutions. As Moser pointed out, "Levinson stands at the crossroad between the earlier classical theory of dynamical systems and the later geometric developments." [14]

Levinson made fundamental contributions to inverse scattering theory by establishing the connection between scattering data and spectral data. In particular, he related the Gel'fand-Levitan method for recovering the potential from the spectrum to the inverse scattering problem for the Schrödinger equation [18]. In the *Selected Papers of Norman Levinson*, D. H. Sattinger remarked on Levinson's work on inverse problems: "The Gel'fand-Levitan method for the inverse Sturm-Liouville problem was applied to the inverse scattering problem...by Levinson in [19]. The treatment in [19] gives a marvelously concise but complete account of the Gel'fand-Levitan method."

In [20], Levinson considered the inverse Sturm-Liouville problem whereby it is assumed that the boundary conditions and eigenfunctions of

$$y' + (1 - P(x))y = 0 \text{ on } [0, \pi]$$

are known and it is required to find the potential $P(x)$. Specifically, he assumed that the potential is Lebesgue integrable and that the boundary conditions are

$$y(0) \cos(a) + y'(0) \sin(a) = y(\pi) \cos(b) + y'(\pi) \sin(b) = 0$$

$$y(0) \cos(a) + y'(0) \sin(a) = y(\pi) \cos(c) + y'(\pi) \sin(c) = 0.$$

He proved that if the spectrum for each of the boundary conditions is given and $\sin(c - b) \neq 0$, then $P(x)$ is uniquely determined. Moreover, he adapted the methods used in [18] to show that if $P(\pi - x) = P(x)$ almost everywhere and $a + b = \pi$, then the spectrum for one set of boundary conditions suffices to determine the potential.

A natural extension of Levinson's work on inverse problems is the study of the spectra of singular self-adjoint second order differential operators of the form

$$Lu = - (pu')' + qu,$$

where p, p' , and q are real and continuous functions and $p > 0$. The problem is said to be singular if the interval on which it is considered is either semi-infinite or infinite—or, in the case of a finite interval, if the coefficients are badly behaved at one or both of the end points. Actually, under suitable restrictions the latter case can be transformed into the former case.

In the case of a semi-infinite interval, consider the problem

$$Lu = \lambda u,$$

where λ is an arbitrary complex number. According to H. Weyl [21], if every solution of the problem is square integrable on the interval, then L is said to be of limit circle type at infinity; otherwise it is said to be of limit point type. This classification depends only on the differential operator L and not on the complex constant λ . In the limit point case, there is exactly one square integrable solution to the differential equation for any λ . In [22] Levinson derived very useful criteria for the limit point case. In other papers he considered expansion theorems and the spectra of singular operators, and he extended his results on expansion theorems to self-adjoint differential operators of arbitrary order.

Many of Levinson's results, along with a very elegant presentation of classical material, appeared in his 1955 book, *Theory of Ordinary Differential Equations*, written in collaboration with E. A. Coddington. This book has been the definitive graduate text in the subject for many generations, and numerous researchers in differential equations learned their trade from it.

Another extremely important aspect of Levinson's research was his work on singularly perturbed systems of differential equations. In a very influential paper [23], he considered the system

$$x' = fu' + k, \quad \varepsilon u'' + gu' + h = 0$$

for $\varepsilon > 0$, where f, k are given n -vectors and g, h are scalar functions of x, u, t , and ε . For $\varepsilon = 0$ the system degenerates to

$$y' = fv' + k, \quad gv' + h = 0$$

and the problem is to find the relationship between solutions of the full system and those of the degenerate system. Problems of this sort arise in many applications. The important feature of these systems is that g is allowed to vanish at a finite number of points so that solutions of the degenerate system may have discontinuities at these points. This occurs, for example, in the rescaled unforced Van der Pol equation

$$\varepsilon u'' + (u^2 - 1)u' + u = 0,$$

which may exhibit discontinuous relaxation oscillations.

A pair (y, v) is said to be a discontinuous solution of the degenerate system if it satisfies the system in the ordinary sense on the open intervals between zeros of g . It is assumed that $g > 0$ on the intervals between the zeros of g and at the end points of $[a, b]$. Finally, the jumps at the points of discontinuity are determined by certain technical conditions. Given a discontinuous solution of the degenerate system on the interval $[a, b]$ with initial conditions $y(a), v(a)$, then for ε and δ sufficiently small the solution $(x(t, \varepsilon), u(t, \varepsilon))$ of the full system with initial values $(x(a), u(a), u'(a))$ exists on $[a, b]$ provided that $(x(a), u(a))$ are sufficiently close to $(y(a), v(a))$ and $|u'(a) - v'(a)| \leq \delta$. Moreover, outside of arbitrarily small neighborhoods of the singular points, the solution $(x(t, \varepsilon), u(t, \varepsilon))$ converges to $(y(t), v(t))$ uniformly as ε and δ tend to zero, and the initial conditions converge. If one also excludes an arbitrarily small right neighborhood of the initial point $t = a$, a similar result holds for the first and second derivatives of u and v . In particular,

the solution to the full system exhibits internal layers in the neighborhood of the zeros of g , but no boundary layers. Extensive further development of these ideas took place in the Soviet Union.[24]

An important special case of the theory developed in [23] concerns the existence of periodic solutions of the full system for small ε , in case f, k, g, h depend explicitly on t , are periodic of period T , and the degenerate problem possesses a possibly discontinuous periodic solution (also of period T). If the periodic solution of the degenerate problem is asymptotically stable, then for ε sufficiently small the full system has a unique periodic solution of period T that also is asymptotically stable.

In collaboration with his former student J. J. Levin, Levinson studied the relationship between solutions of the full system

$$x' = f(x, y, t, \varepsilon), \quad \varepsilon y' = g(x, y, t, \varepsilon)$$

and the degenerate system

$$x' = f(x, y, t, 0), \quad 0 = g(x, y, t, 0),$$

where f is an n -vector and g is an m -vector [25]. They assumed that the degenerate system has a $C^1[a, b]$ solution $x = \varphi(t)$, $y = \psi(t)$ and that f, g are also of class $C^1[a, b]$ in a neighborhood of $\varphi(t)$, $\psi(t)$ for t in $[a, b]$ and $\varepsilon \geq 0$ sufficiently small. The essential hypothesis is that the eigenvalues of the Jacobian matrix of g are all strictly negative. Under these conditions the full system has a unique solution with initial values $x(a) = \zeta$, $y(a) = \eta$ that exists on $[a, b]$, provided that ε is sufficiently small and ζ, η are sufficiently close to $\varphi(a)$, $\psi(a)$ respectively. Moreover, this solution converges to the solution of the degenerate system as ε tends to zero and its initial values converge to those of the solution of the degenerate system. If $x(a, \varepsilon) = \varphi(a)$ but $y(a, \varepsilon)$ is a constant that is different from but close to $\psi(a)$, then the convergence is uniform on $[c, b]$ for any $c > a$. With a slight strengthening of the Jacobian condition, the y of the solution is approximately given by the solution of the boundary layer equation

$$\varepsilon Y' = g(\varphi(a), Y, a, 0), \quad Y(a) = \eta$$

and the width of the boundary layer can be estimated. In joint work with his former student L. Flatto [26], Levinson dealt with the case in which f and g are periodic in t with a period independent of ε . Subsequent developments of the theory outlined here are discussed in R. E. O'Malley's 1991 book [27].

Christopher K. R. T. Jones, in a letter to the editors of the *Selected Papers of Norman Levinson*, states:

like Fenichel...I do see this modern geometric theory as the natural offspring of the classical work of Levinson and others on this topic. The kind of estimates they get for solutions that jump between fast and slow segments—as seen, say, in Levinson’s paper [23]—find a natural expression in the geometric theory.

In a somewhat different direction, Coddington and Levinson [29] studied the existence, uniqueness, and limit behavior of the two-point boundary value problem

$$\varepsilon y'' + f(x, y)y' + g(x, y) = 0, \quad y(0) = a, \quad y(1) = b,$$

where $\varepsilon > 0$ is small and a, b are arbitrary real numbers. A corresponding degenerate problem is

$$f(x, u)u' + g(x, u) = 0, \quad g(1) = b.$$

Here the primes denote differentiation with respect to x . They assumed that the degenerate problem has a unique solution $u(x)$ on $[0, 1]$ with $u(0) > a$, if f and g are smooth in a suitable neighborhood of $u(x)$ and if $f(x, y)$ is positive and bounded away from zero. They showed that for sufficiently small $\varepsilon > 0$, the solution $y(x, \varepsilon)$ of the full problem is unique and converges uniformly to $u(x)$ as ε tends to zero on any subinterval $[c, 1]$ with $c > 0$. It is known that this result is no longer true if $f(x, y)y' + g(x, y)$ is replaced by $F(x, y, y', \varepsilon)$ without further assumptions. Levinson, together with his former student S. Haber [30], derived appropriate conditions on F . These papers inspired a great deal of subsequent work, which was summarized in O'Malley's book [27].

In addition to his work on singular perturbation problems for ordinary differential equations, Levinson wrote a groundbreaking paper [31] on a singular perturbation of the Dirichlet problem for the linear elliptic partial differential equation

$$\varepsilon \Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$$

in an open bounded connected region R of the plane whose boundary G consists of a finite number of simple closed curves. The given functions A, B, C, D are assumed to be sufficiently smooth in a connected open region S that contains $R + G$. For the Dirichlet problem, a smooth boundary function h is given on G . In order to state

his result, Levinson defined a regular quadrilateral Q in $R + G$ as a closed simply connected region bounded by two characteristics of the degenerate first-order equation

$$A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$$

and two sub-arcs G_1 and G_2 of G . For simplicity assume that $A^2 + B^2 > 0$ so that the vector field contains no singular points and S contains no limit cycles. It is further assumed that the characteristics intersect the arcs G_1 and G_2 transversely. The Dirichlet problem for the full equation has a solution $u(x, y, \varepsilon)$, which is uniformly bounded in $R + G$ for all sufficiently small ε . In Q this solution has the representation

$$u(x, y, \varepsilon) = U(x, y) + z(x, y, \varepsilon) + w(x, y, \varepsilon).$$

Here U is the solution of the degenerate equation with boundary values h on G_1 ; It is the main term in the representation of u , except in a thin layer near G_2 (the boundary layer) where the term $z(x, y, \varepsilon)$ dominates. The construction of the boundary layer term z essentially follows a formal procedure that is standard in the theory of singularly perturbed ordinary differential equations. In the boundary layer

$$z(x, y, \varepsilon) = k(x, y) \exp\left(\frac{-g(x, y)}{\varepsilon}\right)$$

$g = 0$ on G_2 , $g > 0$ in Q off G_2 , and g, k are smooth functions that are obtained by solving certain differential equations. There exists a $\delta > 0$ such that

$$z = 0 \left(\exp\left(\frac{-\delta}{\varepsilon}\right) \right)$$

outside the boundary layer. Finally, it is shown that the error term $w = O(\sqrt{\varepsilon})$ and that the width of the boundary layer along G_2 is $O(\varepsilon)$.

Levinson's work was the source of a great deal of further research. For example, S. Kamenomostskaya (Kamin) and Levinson's former student V. Mizel extended his results to the case in which the transversality condition is violated; another former student, D. G. Aronson, extended his results to linear parabolic equations; and his work on singular perturbation problems for partial differential equations was the basis of many subsequent treatments, as shown in [32], [33], and [34]. The work of Levinson on singular perturbations of elliptic partial differential equations also led to the development of a large body

of probabilistic research on related problems, which was summarized in M. I. Freidlin's lecture notes [35].

In addition to the work outlined above, Levinson also studied eigenvalue problems for semi-linear elliptic equations, and he addressed the Dirichlet problem for the equation $\Delta u + f(x, y, u) = 0$ [36]. His approach to these problems was based on the direct method of the calculus of variations and avoided the use of the maximum principle. Thus his approach can be extended to higher-order equations, but this does not seem to have been done; subsequent developments have generally been based on nonlinear functional analysis. He also derived some useful one-sided inequalities for the solutions of second-order elliptic equations.

Levinson made a number of noteworthy contributions to the theory of nonlinear Volterra integral equations, and his results led to subsequent work by his former student J. A. Nohel as well as by Nohel's students and collaborators. Recent developments along these lines were surveyed in the book of G. Gripenberg, S.-O. Londen, and O. Staffans [37].

In his remarks on Levinson's contributions to stochastic analysis, Mark Pinsky wrote: "As a classical analyst par excellence Levinson was in a strong position to bring new light to various questions in probability theory and stochastic processes" [38]. An early essay in this direction was Levinson's work on Wiener's theory of prediction and filtering. In the introduction to his paper, Levinson wrote: "[A] method will be presented for determining quantitatively the extent to which message and noise can be separated. Also will be given a method of designing a filter to carry out this separation. The close of this article will consider the problem of filtering and prediction simultaneously. The root-mean-square error approach used here is an approximation to and a simplification of the transcendental case developed by N. Wiener"[39].

In two papers, [40] and [41], Levinson attacked the problem of the limiting behavior of classical and age-dependent Galton-Watson branching processes. In particular, the question of interest was the occurrence or nonoccurrence of process extinction. This work has had far-reaching influence in the field, as documented in the book of T. E. Harris[42].

In their monumental paper [43], Levinson and H. P. McKean, Jr., studied the relationship between the past and the future of a one-dimensional stationary Gaussian process x of mean 0, with special attention to the degree of dependence on the past as

reflected in the projection of $x(t) : t \geq 0$ upon $x(t) : t \leq 0$. Although the subject was probabilistic, the paper was a tour de force in classical analysis. Its results figured prominently in further work of McKean with H. Dym. In particular, their book *Gaussian Processes, Function Theory, and the Inverse Spectral Problem* was “dedicated to the memory of Norman Levinson, who got us started in this business.”

In its simplest form the prime number theorem states that

$$\pi(x) \sim \frac{x}{\ln(x)}$$

where $\pi(x)$ is equal to the number of primes that are less than or equal to x . This theorem was conjectured by Gauss and Legendre before 1800, but it was not until 1896 that Hadamard and de la Vallée Poussin independently proved it. Their proofs were not elementary in that they involved complex variable methods. In a 1921 lecture in Copenhagen, G. H. Hardy stated:

No elementary proof of the prime number theorem is known, and one may ask whether it is reasonable to expect one. A proof...not fundamentally dependent on the theory of functions seems to me extraordinarily unlikely.

Nevertheless, in 1948 Paul Erdős and Alte Selberg independently published elementary proofs based on simple properties of the logarithm function. Note that an elementary proof was not necessarily an easy one. Indeed the proofs of Erdős and Selberg were very difficult. In 1969 Levinson published “A Motivated Account of an Elementary Proof of the Prime Number Theorem” in the *American Mathematical Monthly* [44], based on Selberg’s approach. This paper was awarded the American Mathematical Association’s 1971 Chauvenet Prize for outstanding exposition.

The Riemann hypothesis has to do essentially with the problem of estimating the error in the prime number theorem approximation. Riemann reduced the problem to finding the roots of the complex zeta function, in particular conjecturing that all of these roots lie on the vertical line $\sigma = 1/2$ in the complex plane. Hardy was apparently the first to prove that infinitely many of the zeros lie on this critical line, and many mathematicians tried to improve on this result without making significant progress. But in 1974, while already suffering from the brain tumor that would ultimately kill him, Levinson published an absolutely astounding paper [45], “More than One Third of the Zeros of the Riemann

Zeta-Function are on $\sigma = 1/2$.” His methods were classical, and the proof required all of his formidable powers as an analyst. Brian Conrey pointed out in [46] that Levinson’s proof also revealed the fact that the $1/3$ of the zeros on the critical line are simple. “In particular,” Conrey said, “this was the first proof that infinitely many of the nontrivial zeros of the zeta-function are simple.” Conrey [47] refined Levinson’s approach to show that at least 40 percent of the zeros are simple and on the critical line. Levinson had suggested a method for significantly improving his result, but he did not live long enough to pursue it and no one else has so far been able to carry it through.

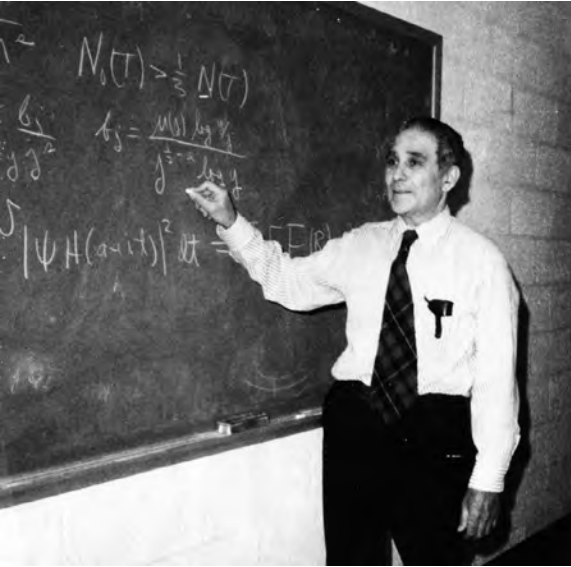
In 1972 Levinson published “ *Ω -Theorems for the Riemann Zeta-Function*” [48]. According to Conrey [49] “Omega theorems give lower bounds for how large the extreme values of a given function can be. Levinson proved the best omega theorems, at the time, for the zeta-function in the strip $1/2 \leq \sigma \leq 1$.” This paper of Levinson stimulated further work in the area, particularly by H. L. Montgomery [50] and K. Ramachandra [51], who sharpened Levinson’s omega results on the critical line.

Levinson was extremely prolific. In addition to the research outlined above, he wrote on many other topics, including control theory and linear programming. In all he had 124 publications, among which were three books. He had 34 Ph.D. students and, according to the Mathematical Genealogy Project, at least 462 descendants. His honors included a National Research Council Fellowship (1935–37), a Guggenheim Fellowship (1948–1949), the Bôcher Prize of the American Mathematical Society (1954), service as vice-president of the American Mathematical Society (1965), election to the National



Levinson, right, with Earl Coddington at the International Congress of Mathematicians, Moscow, 1966.

(Photo courtesy MIT Museum.)



Undated photo.
(Photo courtesy MIT Museum.)

Academy of Sciences (1967), and the Chauvenet Prize of the Mathematical Association of America (1971). Levinson was head of the MIT Mathematics Department from 1968 to 1971, and in that latter year he was appointed Institute Professor.

Together with W. T. Martin, C. C. Lin, and I. M. Singer, Levinson played a leading role in transforming the MIT Mathematics Department from a service department to the extremely strong research entity that it is today. He had a keen eye for mathematical talent and very catholic taste. In 1968, when Martin retired as head of the department, there was a widening gap between the pure and applied mathematicians; the applied people wanted to split off and form a separate department. But the MIT

administration was unwilling to allow such a split. The solution was the formation of an applied group and a pure group, each with its own standards for appointment and promotion. There would also be a Departmental Council, composed of some members from each group along with the department chairman and others, which would review the recommendations of the two groups and transmit the agreed results to the Institute Science Council for final approval. In order to make this system workable it was decided to limit the term of the Department Head to five years (Martin had served for 21 years) and have the leadership alternate between the two groups. "So who would be the first chairman under the new arrangement?" said A. P. Mattuck. "Levinson had credentials in both camps. [He] had both status and broad appeal, so he was the natural choice. [He] served for three years." [52]

Aside from their importance to the scientific aspects of life in the MIT Mathematics Department, the Levinsons were an integral part of its social fabric. Among other things, they ministered to Norbert Wiener during his frequent bouts of depression, helped

improve life for Felix Browder during his time in the Army, and provided important support for John Nash and his wife during his prolonged illness. Fagi, who was characterized as the “den mother” of the department, died in 2009. Norman died on October 10, 1975.

On December 17, 1975, the MIT faculty voted the following memorial resolution:

Norman Levinson was the heart of mathematics at MIT, a man who combined creative intellect of the highest order with human compassion and unremitting devotion to science and to excellence in its pursuit. Throughout the mathematical world the name MIT and the name Norman Levinson have been synonymous for many years. For those of us who were fortunate to have him as a friend and colleague, this is entirely fitting, because we are aware that, with extraordinary effectiveness and caring, he devoted 46 years of his life to mathematics and this institute.
[53]

SOURCES

Except where explicitly noted, most of the biographical information in this article was adapted from the biography of Norman Levinson at www-history.mcs.st-and.ac.uk/Biographies/Levinson.html, written by J. J. O’Connor and E. F. Robertson. Additional biographical information and some of the scientific information came from *Recountings: Conversations with MIT Mathematicians*, edited by Joel Segal. The indispensable resource for the biographical and scientific aspects of Levinson’s career is the two-volume set *Selected Papers of Norman Levinson* edited by John A. Nohel and David H. Sattinger. I am greatly indebted to Peter Duren, Linda Sons, Andrew Odlyzko, and Juan Luis Vazquez for their help.

NOTES

- [1] Nohel, J. A., and D. H. Sattinger (eds.). 1997. Remarks by William T. Martin in *Selected papers of Norman Levinson*. pp. xxxii-xxxv. Boston-Basel-Berlin: Birkhäuser.
- [2] Segal, J. (ed.). 2009. Interview with Zipporah (Fagi) Levinson in *Recountings: Conversations with MIT mathematicians*. p. 5. Wellesley MA: A. K. Peters, Ltd.
- [3] U.S. House of Representatives, Un-American Activities Committee. 1953. *Communist methods of Infiltration (Education—Part 4)*. Hearings of April 23. p. 1101. Washington, DC: U. S. Government Printing Office.
- [4] Levinson, N. 1946. The asymptotic behavior of systems of linear differential equations. *American J. Math.* 68:1–6.
- [5] Levinson, N. 1948. The asymptotic nature of solutions of linear systems of differential equations. *Duke Math. J.* 15:111–126.
- [6] Pego, R., and M. Weinstein. 1992. Eigenvalues and solitary wave instabilities. *Phil. Trans. R. Soc. London* 340A:47–94.
- [7] Levinson, N. 1949. On stability of nonlinear systems of differential equations. *Colloquium Math.* 2:40–45.
- [8] Bellman, R. 1947. On the boundedness of solutions of nonlinear differential and difference equations. *Trans. Amer. Math. Soc.* 62:347–386.
- [9] Van der Pol, B. 1926. Relaxation oscillations. *Philosophical Magazine, Seventh Series* 2:978.
- [10] Liénard, A. 1928. Etude des oscillations entretenues. *Revue Gén. de l'Electricité* 33:901–946.
- [11] Levinson, N., and O. K. Smith. 1942. A general equation for relaxation oscillations. *Duke Math. J.* 9:382–403.
- [12] Cartwright, M. L., and J. E. Littlewood. 1945. On nonlinear differential equations of second order. I. The equation $\ddot{y} + k(1 - y^2)y + \dot{y} = b\lambda k \cos(\lambda t + a)$, k large. *J. London Math. Soc.* 20:180–189.
- [13] Levinson, N. 1949. A second-order differential equation with singular solutions. *Ann. of Math.* 50:127–153.
- [14] Nohel, J. A., and D. H. Sattinger (eds.). 1997. Remarks by Jürgen Moser in Volume 1 of *Selected papers of Norman Levinson*. Boston-Basel-Berlin: Birkhäuser.
- [15] Smale, S. 1960. On dynamical systems. *Bol. Soc. Math. Mexicana* 2:195–198.
- [16] Smale, S. 1963. How I got started in dynamical systems, 1959–1962. In *From Topology to Computation: Proceedings of the SMALFEST*. Edited by W. M. Hirsh, J. E. Marsden, and M. Shub. pp. 22-26. New York: Springer-Verlag.

- [17] Levi, M. 1981. *Qualitative analysis of the periodically forced relaxation oscillations*. Providence, RI: Memoirs of the American Mathematics Society 32.
- [18] Levinson, N. 1949. On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase. *Danske Vid. Selsk. Mat.-Fys. Medd.* 25:1–29.
- [19] Levinson, N. 1953. Certain explicit relationships between phase shift and scattering potential. *Phys. Review* 89:755–757.
- [20] Levinson, N. 1949. The inverse Sturm-Liouville problem. *Mat. Tidsskr. B.* 1949:25–30.
- [21] Weyl, H. 1909. Über gewöhnliche lineare differentialgleichungen mit singulären stellen und ihre eigenfunctionen. *Nachr. Ges. Wiss. Göttingen* 1909:37–64; Weyl, H. 1910. Über gewöhnliche lineare differentialgleichungen mit singulären stellen und die zugehörigen entwicklungen willkürlichen funktionen. *Math Ann.* 68:220–269; Weyl, H. 1910. Über gewöhnliche differentialgleichungen mit singulären stellen und ihre eigenfunctionen. *Nachr. Ges. Wiss. Göttingen* 1910:442–467.
- [22] Levinson, N. 1949. Criteria for the limit-point case for second-order linear differential operators. *Casopis. Pest. Mat. Fys.* 74:17–20.
- [23] Levinson, N. 1950. Perturbations of discontinuous solutions of non-linear systems of differential equations. *Acta Math.* 82:71–106.
- [24] Miscenco, E. F., and N. Kh. Rozov. 1980. *Differential equations with small parameters and relaxation oscillations*. Translated by F. M. C. Goodspeed. New York: Plenum Publishers.
- [25] Levin, J. J., and N. Levinson. 1954. Singular perturbations of nonlinear systems of differential equations and an associated boundary layer equation. *J. Rat. Mech. Anal.* 3:247–256.
- [26] Flatto, L., and N. Levinson. 1955. Periodic solutions of singularly perturbed systems. *J. Rat. Mech. Anal.* 4:943–950.
- [27] O'Malley, Jr., R. E. 1991. *Singular perturbation methods for ordinary differential equations*. Applied Mathematical Sciences, Vol. 89. New York: Springer-Verlag.
- [28] Nohel, J. A., and D. H. Sattinger (eds.). 1997. Remarks by John A. Nohel in Volume 1 of the *Selected papers of Norman Levinson*. p. 227. Boston-Basel-Berlin: Birkhäuser.
- [29] Coddington, E. A., and N. Levinson. 1952. A boundary value problem for a nonlinear differential equation with a small parameter. *Proc. Amer. Math. Soc.* 3:73–81.
- [30] Haber, S., and N. Levinson. 1955. A boundary value problem for a singularly perturbed differential equation. *Proc. Amer. Math. Soc.* 6:866–872.
- [31] Levinson, N. 1950. The first boundary value problem for $\epsilon\Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$ for small ϵ . *Ann. of Math.* 51:428–445.

- [32] Vishik, M. I., and L. A. Lyusternik. 1957. Regular degeneration and boundary layer for linear differential equations with a small parameter. *Usp. Mat. Nauk* 12:3–122 (Russian); Vishik, M. I., and L. A. Lyusternik. 1961. *Amer. Math. Soc. Transl. Ser. 2*:239–364.
- [33] Eckhaus, W., and E. M. De Jager. 1966. Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type. *Archive Rat. Mech. Anal.* 23:26–86.
- [34] Lions, J. L. 1973. Perturbations singulières dans les problèmes aux limites et en contrôle optimal. *Lecture Notes in Math. Vol. 323*. Berlin-Heidelberg: Springer-Verlag.
- [35] Freidlin, M. I. 1996. Markov processes and differential equations: Asymptotic problems. *Lectures in Mathematics of ETH Zürich*. Basel-Boston-Berlin: Birkhäuser.
- [36] Levinson, N. 1963. Dirichlet problem for $\Delta u = F(P, u)$. *J. Math. and Mech.* 12:440–448.
- [37] Gripenberg, G., S.-O. London, and O. Staffans. 1990. Volterra integral and functional equations. In *Encyclopedia of mathematics and its applications, Vol. 34*. Cambridge, UK: Cambridge University Press.
- [38] Nohel, J. A., and D. H. Sattinger (eds.). 1997. Remarks by Mark Pinsky in Volume 2 of the *Selected papers of Norman Levinson*. pp. 160-161. Boston-Basel-Berlin: Birkhäuser.
- [39] Levinson, N. 1947. The Wiener RMS (root mean square) error criterion in filter design and prediction. *J. Math. Phys. Mass. Inst. Tech.* 25:261–278.
- [40] Levinson, N. 1959. Limiting theorems for Galton-Watson branching process. *Illinois J. Math.* 3:554–565.
- [41] Levinson, N. 1960. Limiting theorems for age-dependent branching process. *Illinois J. Math.* 4:110–118.
- [42] Harris, T. E. 1963. Branching processes. *Grundlehren der Mathematischen Wissenschaften*. Berlin-Göttingen-Heidelberg: Springer-Verlag.
- [43] Levinson, N., and H. P. McKean, Jr. 1964. Weighted trigonometrical approximations on \mathbb{R}^1 with applications to the germ field of a stationary Gaussian noise. *Acta Math.* 112:99–143.
- [44] Levinson, N. 1969. A motivated account of an elementary proof of the prime number theorem. *Amer. Math. Monthly* 76:225–245.
- [45] Levinson, N. 1974. More than one-third of the zeros of Riemann's zeta-function are on $\sigma = 1/2$. *Advances in Math.* 13:383–436.
- [46] Nohel, J. A., and D. H. Sattinger (eds.). 1997. Remarks by Brian Conrey in Volume 2 of the *Selected Papers of Norman Levinson*. pp. 306-310. Boston-Basel-Berlin: Birkhäuser.
- [47] Conrey, J. B. 1989. More than two-fifths of the zeros of the Riemann zeta-function are on the critical line. *J. Reine Angew. Math.* 399:1–26.

- [48] Levinson, N. 1972. Ω -theorems for the Riemann zeta-function. *Acta Arith.* 20:317–330.
- [49] Nohel, J. A., and D. H. Sattinger (eds.). 1997. Remarks by Brian Conrey in Volume 2 of the *Selected Papers of Norman Levinson*. pp. 423-424. Boston-Basel-Berlin: Birkhäuser.
- [50] Montgomery, H. L. 1977. Extreme values of the Riemann zeta-function. *Comment. Math. Helv.* 52:511–518.
- [51] Ramachandra, K. 1995. On the mean value and omega-theorems for the Riemann zeta-function. *Tata Institute of Fundamental Research Lectures on Mathematics and Physics* 85. Berlin: Springer-Verlag.
- [52] Segal, J. (ed.). 2009. Interview with Arthur P. Mattuck in *Recountings: Conversations with MIT mathematicians*. Wellesley, MA: A. K. Peters Ltd.
- [53] Nohel, J. A., and D. H. Sattinger (eds.). 1997. Biographical sketch of Norman Levinson in Volume 1 of the *Selected Papers of Norman Levinson*. p. xvii. Boston-Basel-Berlin: Birkhäuser.

SELECTED BIBLIOGRAPHY

- 1938 General gap Tauberian theorems: I. *Proc. London Math. Soc.* 14:289–306.
- 1940 *Gap and density theorems*, Vol. 26. New York: American Mathematical Society Colloquium Publications.
- 1942 With O. K. Smith. A general equation for relaxation oscillations. *Duke Math. J.* 9:382–403.
- 1946 The asymptotic behavior of a system of differential equations. *American J. Math.* 68:1–6.
- 1949 A second-order differential equation with singular solutions. *Ann. of Math.* 50:127–153.
- Criteria for the limit-point case for second-order differential operators. *Casopis Pest. Mat. Fys.* 74:17–20.
- On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase. *Danske Vid. Selsk. Mat.-Fys. Medd.* 25:1–29.
- The inverse Sturm-Liouville problem. *Mat. Tidsskr. B.* 1949:25–30.
- 1950 Perturbations of discontinuous solutions of nonlinear systems of differential equations. *Acta Math.* 82:71–106.
- The first boundary value problem for $\varepsilon\Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$ for small ε , *Ann. of Math.* 51:428–445.
- 1953 Certain explicit relationships between phase shift and scattering potential. *Phys. Rev.* 89:755–757.
- 1954 The expansion theorem for singular self-adjoint differential operators. *Ann. of Math.* 59:300–315.
- With J. J. Levin. Singular perturbations of nonlinear systems of differential equations and an associated boundary layer equation. *J. Rational Mech. Anal.* 3:247–270.

- 1955 With E. A. Coddington. *Theory of ordinary differential equations*. New York-Toronto-London: McGraw Hill.
- With L. Flatto. Periodic solutions of singularly perturbed systems. *J. Rational Mech. Anal.* 4:943–950.
- With S. Haber. A boundary value problem for a singularly perturbed differential equation. *Proc. Amer. Math. Soc.* 6:866–872.
- 1959 Limiting theorems for Galton-Watson branching process. *Illinois J. Math.* 3:554–565.
- 1960 Limiting theorems for age-dependent branching process. *Illinois J. Math.* 4:100–118
- A nonlinear Volterra equation arising in the theory of superfluidity. *J. Math. Anal. Appl.* 1:1–11.
- 1962 Positive eigenfunctions for $\Delta u + \lambda f(u) = 0$. *Arch. Rational Mech. Anal.* 11:258–272.
- 1964 With H. P. McKean, Jr. Weighted trigonometrical approximations on \mathbb{R}^1 with applications to the germ field of a stationary Gaussian noise. *Acta Math.* 112:99–143.
- 1969 A motivated account of an elementary proof of the prime number theorem. *Amer. Math. Monthly* 76:225–245.
- 1972 Ω -theorems for the Riemann zeta-function. *Acta Arith.* 20:317–330.
- 1974 More than one-third of the zeros of Riemann's zeta-function are on $\sigma = 1/2$. *Advances in Math.* 13:383–486.
- With H. L. Montgomery. Zeros of the derivatives of the Riemann zeta-function. *Acta Math.* 133:49–65.

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