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LOO-KENG HUA

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A Biographical Memoir by

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Loo-Keng Hua was one of the leading mathematicians of his time and one of the two most eminent Chinese mathematicians of his generation, S. S. Chern being the other. He spent most of his working life in China during some of that country's most turbulent political upheavals. If many Chinese mathematicians nowadays are making distinguished contributions at the frontiers of science and if mathematics in China enjoys high popularity in public esteem, that is due in large measure to the leadership Hua gave his country, as scholar and teacher, for 50 years.

Hua was born in 1910 in Jintan in the southern Jiangsu Province of China. Jintan is now a flourishing town, with a high school named after Hua and a memorial building celebrating his achievements; but in 1910 it was little more than a village where Hua's father managed a general store with mixed success. The family was poor throughout Hua's formative years; in addition, he was a frail child afflicted by a succession of illnesses, culminating in typhoid fever that caused paralysis of his left leg; this impeded his movement quite severely for the rest of his life. Fortunately Hua was blessed from the start with a cheerful and optimistic disposition, which stood him in good stead then and during the many trials ahead.

Hua's formal education was brief and, on the face of it, hardly a preparation for an academic career—the first degree he would receive was an honorary doctorate from the University of Nancy in France in 1980; nevertheless, it was of a quality that did help his intellectual development. The Jintan Middle School that opened in 1922 just when he had completed elementary school had a well-qualified and demanding mathematics teacher who recognized Hua's talent and nurtured it. In addition, Hua learned early on to make up for the lack of books, and later of scientific literature, by tackling problems directly from first principles, an attitude that he maintained enthusiastically throughout his life and encouraged his students in later years to adopt.

Next, Hua gained admission to the Chinese Vocational College in Shanghai, and there he distinguished himself by winning a national abacus competition; although tuition fees at the college were low, living costs proved too high for his means and Hua was forced to leave a term before graduating. After failing to find a job in Shanghai, Hua returned home in 1927 to help in his father's store. In that same year also, Hua married Xiaoguan Wu; the following year a daughter, Shun, was born and their first son, Jundong, arrived in 1931.

By the time Hua returned to Jintan he was already engaged in mathematics and his first publication, "Some Researches on the Theorem of Sturm," appeared in the December 1929 issue of the Shanghai periodical *Science*. In the following year Hua showed in a short note in the same journal that a certain 1926 paper claiming to have solved the quintic was fundamentally flawed. Hua's lucid analysis caught the eye of a discerning professor at Quing Hua University in Beijing, and in 1931 Hua was invited, despite his lack of formal qualification and not without some reservations on the part of several faculty members, to join the mathematics department there. He began as a clerk in the library, and then moved to become an assistant in mathematics; by September 1932 he was an instructor and two years later came promotion to the rank of lecturer. By that time he had published another dozen papers and in some of these one could begin to find intimations of his future interests; thanks to his natural talent and dedication, Hua was now, at the age of 24, a professional mathematician.

At this time Quing Hua University was the leading Chinese institution of higher education, and its faculty was in the forefront of the endeavor to bring the country's mathematics and science abreast of knowledge in the West, a formidable task after several hundred years of stagnation. During 1935-36 Hadamard and Norbert Wiener visited the university; Hua eagerly attended the lectures of both and created a good impression. Wiener visited England soon afterward and spoke of Hua to G. H. Hardy. In this way Hua received an invitation to come to Cambridge, England, and he arrived in 1936 to spend two fruitful years there. By now he had published widely on questions within the orbit of Waring's problem (also on other topics in diophantine analysis and function theory) and he was well prepared to take advantage of the stimulating environment of the Hardy-Littlewood school, then at the zenith of its fame. Hua lived on a $1,250 per annum scholarship awarded by the Culture and Education Foundation of China; it is interesting to recall that this foundation derived its funds from reparations paid by China to the United States following wars waged in China by the United States and several other nations in the previous century. The amount of the grant imposed on him a Spartan regime. Hardy assured Hua that he could gain a Ph.D. in two years with ease, but Hua could not afford the registration fee and declined; of course, he gave quite different reasons for his decision.

During the Cambridge period Hua became friendly with Harold Davenport and Hans Heilbronn, then two young research fellows of Trinity College—one a former student of Littlewood and the other Landau's last assistant in Göttingen—with whom he shared a deep interest in the Hardy-Littlewood approach to additive problems akin to Waring's. They helped to polish the English in several of Hua's papers, which now flowed from his pen at a remarkable rate; more than 10 of his papers date from this time, and many of these appeared in due course in the publications of the London Mathematical Society.

About the only easy thing about Waring's problem is its statement: In 1770 Waring asserted without proof (and not in these words) that for each integer $k \geq 2$ there exists an integer $s = s(k)$ depending only on k such that every positive integer N can be expressed in the form

$$N = x_1^k + x_2^k + \ldots + x_s^k \tag{1}$$

where the $x_i (i = 1,2, \ldots, s)$ are non-negative integers. In that same year Lagrange had settled the case $k = 2$ by showing that $s(2) = 4$, a best possible result; after that, progress was painfully slow, and it was not until 1909 that Hilbert solved Waring's problem in its full generality. His argument rested on the deployment of intricate algebraic identities and yielded rather poor admissible values of $s(k)$. In 1918 Hardy and Ramanujan returned to the case $k = 2$ in order to determine the number of representations of an integer as the sum of s squares by means of Fourier analysis, an approach inspired by their famous work on partitions, and they succeeded. This encouraged Hardy and Littlewood in 1920 to apply a similar method for general k, and they devised the so-called circle method to tackle the general Hilbert-Waring theorem and a host of other additive problems, Goldbach's problem among them. During the next 20 years the machinery of the circle method came to be regarded about as difficult as anything in the whole of mathematics; even today, after numerous refinements and much progress, the intricacies of the method remain formidable.[1] In outline, the circle method of Hardy-Littlewood-Ramanujan, as modified by I.M.Vinogradov, is as follows: Let

$$T(\alpha)=\sum_{x=0}^{P} e^{2\pi i\alpha x^k}, P=\lfloor N^{1/k}\rfloor,$$

and denote by $R_s^{(k)}(N)$ the number of representations of N in the form (1). Then

$$R_s^{(k)}(N)=\int_0^1 T(\alpha)^s e^{-2\pi i N\alpha}d\alpha,$$

and to prove the Hilbert-Waring theorem it is enough to show that $R_s^{(k)}(N)>0$ for all large enough positive integers N, with s some natural number depending only on k. The least admissible value of s is denoted by $G(k)$. The generating function $T(\alpha)$ is well approximable on each of a family of disjoint intervals in [0,1] centered at rational numbers with small denominators, and the plan is to show that the main contribution to $R_s^{(k)}(N)$ comes from integration over these intervals while the integral over the complement, usually referred to as the minor arcs, is of a lesser order of magnitude. The latter task, estimation on the minor arcs, is the harder, but here Hardy and Littlewood used an estimate of a trigonometric sum more general that $T(\alpha)$ that Hermann Weyl had established in 1916 in connection with his fundamental work on criteria for the uniform distribution of a sequence. In this way they were able to show that

$$G(k) \le k2^{k-1} + 1.$$

This is the background against which Hua set to work as a young man, and it is probably fair to say that it is for his contributions in this area that Hua's name will remain best remembered: notably for his seminal work on the estimation of trigonometric sums like $T(\alpha)$, singly or on average. One such average result, now known as Hua's lemma, asserts that for any $\varepsilon > 0$ and for $1 \le j \le k$,

$$\int_0^1 \left|T(\alpha)\right|^{2^j} d\alpha = O_\varepsilon\left(P^{2^j - j + \varepsilon}\right);$$

since $\left|T(\alpha)\right| \le P$ trivially, this estimate yields a saving over the trivial of almost j, for each $j \le k$, in the exponent of P. When used on the minor arcs in conjunction with Weyl's estimate of $T(\alpha)$, Hua's Lemma led to the improved bound

$$G(k) \le 2^k + 1.$$

Nowadays better results are known,[1] but all involve major difficulties.

Hua might well have remained in England longer, but home was never far from his thoughts and the Japanese invasion of China in 1937 caused him much anxiety. He left Cambridge in 1938 to return to his old university, now as a full professor. However, Quing Hua University was no longer in Beijing; with vast portions of China under Japanese occupation, it had migrated to Kunming, the capital of the southern province of Yunan, where it combined with several other institutions to form the temporary Associated University of the South West. There Hua and his family remained through the World War II years, until 1945, in circumstances of poverty, physical privation, and intellectual isolation. Despite these hardships Hua maintained the level of intensity of his Cambridge period and even exceeded it; by the end of 1945 he had more than 70 publications to his name. During this time he studied Vinogradov's seminal method of estimating trigonometric sums and reformulated it, even in sharper form, in what is now known universally as Vinogradov's mean value theorem. This famous result is central to improved versions of the Hilbert-Waring theorem, and has important applications to the study of the Riemann zeta function. Hua wrote up this work in a booklet that was accepted for publication in Russia as early as 1940, but owing to the war, did not appear (in expanded form) until 1947 as a monograph of the Steklov Institute.

Hua spent three months in Russia in the spring of 1946 at Vinogradov's invitation. Mathematical interaction apart, he was impressed by the organization of scientific activity there, and this experience influenced him when later he reached a position of authority in the new China. In the years ahead, even though Hua's scientific activities branched out in other directions, Hua was always ready to return to Waring's problem, to number theory in general and especially to questions involving exponential sums; thus as late as 1959 he published an important monograph on "Exponential Sums and Their Aplications in Number Theory" for the *Enzyklopädie der Matematischen Wissenschaften*. His instinct for what was important and his marvelous command of technique make his papers on number theory even now virtually an index to the major activities in that subject during the first half of the twentieth century.

In the closing years of the Kunming period Hua turned his interests to algebra and to analysis, as much as anything for the benefit of his students in the first instance, and soon began to make original contributions in these subjects too. Thus Hua became interested in matrix algebra and wrote several substantial papers on the geometry of matrices. He had been invited to visit the Institute for Advanced Study in Princeton, but because C. L. Siegel was working there along somewhat similar lines, Hua declined, at first in order to develop his ideas independently. In September 1946, shortly after returning from Russia, Hua did depart for Princeton, bringing with him projects not only in matrix theory but also in functions of several complex variables and in group theory. At this time civil war was raging in China and it was not easy to travel; therefore, the Chinese authorities assigned Hua the rank of general in his passport for the "convenience of travel."

According to his biographer, Hua's "most significant and rewarding research work" during his stay in the United States was on the topic of skew fields, that is, on (non-commutative) division rings, of which the quaternions are a classic example. Thus, Hua was the first to show that every semi-automorphism of a skew field F is either an automorphism or an anti-automorphism—more explicitly, if σ is a one-to-one mapping of F onto itself such that $1^\sigma = 1$ and

$$(a+b)^\sigma = a^\sigma + b^\sigma, \ (aba)^\sigma = a^\sigma b^\sigma a^\sigma,$$

for all a, b in F, then either

$$(ab)^{\sigma} = a^{\sigma} b^{\sigma} \text{ for all } a, b \text{ in } F$$

or

$$(ab)^{\sigma} = b^{\sigma} a^{\sigma}.$$

He also gave a spectacular demonstration of his “direct” approach to problems with his proof that every normal subfield of a skew field is contained in its center. The argument is only one and a half pages long and rests on the following identity: if $ab \neq ba$, then

$$a = \left(b^{-1} - (a-1)^{-1} b^{-1} (a-1)\right)\left(a^{-1} b^{-1} a - (a-1)^{-1} b^{-1} (a-1)\right).$$

This result is now known in the literature as the Cartan-Brauer-Hua theorem; originally H. Cartan’s proof had used much deeper tools.

There was much else, of course, to distinguish this last major creative period of his life. Hua wrote several papers with H. S. Vandiver on the solution of equations in finite fields and with I. Reiner on automorphisms of classical groups. Much of his algebraic work later provided the basis for the monograph “Classical Groups” by Wan Zhe Xian and Hua (published by the Shanghai Scientific Press in Chinese in 1963).

On the personal side, in the spring of 1947 Hua underwent an operation at the Johns Hopkins University on his lame leg that much improved his gait thereafter, to his and his family’s delight. Also in 1947 their daughter Su was born; two more sons had arrived earlier, Ling and Guang, the latter in 1945 and one more daughter, Mi, was born a little later. In the spring of 1948 Hua accepted appointment as a full professor at the University of Illinois in Urbana-Champaign. There he directed the thesis of R. Ayoub, later a professor at Pennsylvania State University; continued his work with I. Reiner; and influenced the thinking of several young research workers, L. Schoenfeld and J. Mitchell among them. His stay in Illinois was all too brief; exciting developments were taking place in China, and Hua watched them eagerly, wanting to be part of the new epoch. Although he had brought his wife and three younger children to Urbana and they had settled in quite well, the urge to return was too great; on March 16, 1950, he was back in Beijing at his alma mater, Quing Hua University, ready to add his contribution to the brave new world. He was then at the peak of his mathematical powers and, as he wrote to me many years later, the 1940s had been to him in retrospect the golden years of his life. Despite the trials that he would face, he did not at any subsequent time regret his decision to return.

Back in China, Hua threw himself into educational reform and the organization of mathematical activity at the graduate level,[2] in the schools, and among workers in the burgeoning industry. In July 1952 the Mathematical Institute of the Academia Sinica came into being, with Hua as its first director. The following year he was one of a 26-member delegation from the Academia Sinica to visit the Soviet Union in order to establish links with Russian science. At this time Hua entertained doubts whether the Communist Party at home trusted him, and it came as an agreeable surprise to him to learn in Moscow that the Chinese government had agreed to a proposal by the Soviet government to award Hua a Stalin Prize. Following Stalin's death the prize was discontinued, and Hua missed out; in view of later developments, he told me, he had a double reason to be satisfied!

Despite his many teaching and administrative duties, Hua remained active in research and continued to write, not only on topics that had engaged him before but also in areas that were new to him or had been only lightly touched on before. In 1956 his voluminous text, *Introduction to Number Theory,* appeared. (The preface to the 1975 Chinese edition was excised by government order because Hua was out of favor during much of the Cultural Revolution); later this was published by Springer in English translation and is still in print. *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains* came out in 1958 and was translated into Russian in the same year, followed by an English translation by the American Mathematical Society in 1963. Most of the results of this important monograph are due to Hua, with some overlap with the work of Siegel. The results have applications to representation theory, the theory of homogeneous spaces, and to the theory of automorphic forms. The monograph also includes joint work with K. H. Look on the Poisson and Bergman kernels. This work was useful in later investigations by E. Stein on boundary behavior of holomorphic functions. Another aspect of this work reveals that by 1959 Hua appreciated the importance of extending Hodge theory to open Hermitian manifolds; J. J. Kohn did this successfully in 1962. The monograph on exponential sums mentioned earlier, which was published in 1959, has also since been translated into English. Hua was a fluent and prolific writer, there being many books and articles by him in Chinese for schools and for undergraduate use to make modern mathematics accessible to students. He also wrote poetry all his life, for his own amusement and for the pleasure of his friends.

In 1958 he suffered a rude awakening from utopian dreams with the so-called Great Leap Forward, when a Mao-inspired, savage assault on intellectuals swept the country, implemented with enthusiasm by a compliant bureaucracy inspired by Orwellian slogans like “the lowliest are the smartest, the highest the most stupid.” Despite his eminence and some protection in high places, Hua had to suffer harassment, public abuse, and constant surveillance. Nevertheless, during this troubled period Hua developed, with Wang Yuan, a broad interest in linear programming, operations research, and multidimensional numerical integration. In connection with the last of these, the study of the Monte Carlo method and the role of uniform distribution led them to invent an alternative deterministic method based on ideas from algebraic number theory. Their theory was set out in *Applications of Number Theory to Numerical Analysis*, which was published much later, in 1978, and by Springer in English translation in 1981. The newfound interest in applicable mathematics took him in the 1960s, accompanied by a team of assistants, all over China to show workers of all kinds how to apply their reasoning faculty to the solution of shop-floor and everyday problems. Whether in ad hoc problem-solving sessions in factories or open-air teachings, he touched his audiences with the spirit of mathematics to such an extent that he became a national hero and even earned an unsolicited letter of commendation from Mao, this last a valuable protection in uncertain times. Hua had a commanding presence, a genial personality, and a wonderful way of putting things simply, and the impact of his travels spread his fame and the popularity of mathematics across the land.[3] When much later he traveled abroad, wherever he stayed Chinese communities of all political persuasions flocked to meet him and do him honor; in 1984 when he organized a conference on functions of several complex variables in Hangzhou, colleagues from the West were astonished by the scale of the publicity accorded it by the Chinese media.

But all that was in the future. In 1966 Mao set in motion the next national calamity, which came to be known as the Cultural Revolution and would last 10 years. A pronouncement of Mao dated as early as June 26, 1965, sent a dire signal of things to come to the intellectuals: “The more you read, the more stupid you become.” Hua spent many of these years under virtual house arrest. He attributed his survival to the personal protection of Chou En-lai. Even so, he was exposed to harassing interrogations, some of his manuscripts (on mathematical economics) were confiscated and are now irretrievably lost, and attempts were made to extract from his associates and former students damaging allegations against him. (In 1978 the Chinese ambassador to the United Kingdom described one such occasion to me; Chen Jing-run, then probably the best known Chinese mathematician of the next generation, was made to stand in a public place for several hours, surrounded by a mob, and exhorted to bear witness against Hua. Chen, present at this conversation, chimed in to say that, actually, he had quite enjoyed the occasion, since no student could trouble him with silly questions and he had had time, uninterrupted, to think about mathematics!) It is surely no accident that the flow of Hua’s publications came to an untimely end in 1965. He continued to work, of course. There are several joint papers on numerical analysis (with Wang Yuan) and on optimization (with Ke Xue Tong Bao) in the 1970s, but these are probably based on work done earlier; there are also expository articles and texts derived from the vast teaching and consulting experience he accumulated over the years. As he would reminisce sadly in a 1991 article, “Upon entering [my] sixtieth year . . . almost all energy and spirit were taken from me.”

With the end of the Cultural Revolution in 1976 Hua entered upon the last period of his life. Honor was restored to him at home, and he became a vice-president of Academia Sinica, a member of the People's Congress and science advisor to his government. In addition, Chinese Television (CCTV) produced a mini-series telling the story of Hua's life, which has been shown at least twice since then. In 1980 he became a cultural ambassador of his country charged with reestablishing links with Western academics, and during the next five years he traveled extensively in Europe, the United States, and Japan. In 1979 he was a visiting research fellow of the then Science Research Council of the United Kingdom at the University of Birmingham and during 1983-84 he was Sherman Fairchild Distinguished Scholar at the California Institute of Technology. For much of this time he was tired and in poor health, but a characteristic zest for life and a quenchless curiosity never deserted him; to a packed audience in a seminar in Urbana in the spring of 1984 he spoke about mathematical economics. One felt that he was driven to make up for all those lost years. In his last letter to me, dated May 21, 1985, he reported that unfortunately most of his time now was devoted to "non-mathematical activities, which are necessary for my country and my people." He died of a heart attack at the end of a lecture he gave in Tokyo on June 12, 1985.

Hua received honorary doctorates from the University of Nancy (1980), the Chinese University of Hong-Kong (1983), and the University of Illinois (1984). He was elected a foreign associate of the National Academy of Sciences (1982) and a member of the Deutsche Akademie der Naturforscher Leopoldina (1983), Academy of the Third World (1983), and the Bavarian Academy of Sciences (1985).

Professor Wang Yuan has written a fine biography of Hua,[4] and I am indebted to it for some of the information I have used. I have also drawn on the obituary notice I wrote for *Acta Arithmetica* (LI(1988):99-117).

NOTES

1. R. C. Vaughan. *The Hardy-Littlewood Method,* 2nd ed. Cambridge: Cambridge University Press, 1997.

2. Among his students were Chen Jing-run, Pan Chen-dong, and Wang Yuan in number theory; Wan Zhi Xian in algebra; and Kung Sheng and Lu Qi Keng in analysis.

3. For a selection of the problems dealt with, see *Popularizing Mathematical Methods in the People's Republic of China*, by L.-K. Hua and Y. Wang. Boston: Birkhäuser, 1989.

4. Hua Loo-keng. Translated by Peter Shiu. Singapore: Springer, 1999.

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