NATIONAL ACADEMY OF SCIENCES

EINAR HILLE

1894—1980

A Biographical Memoir by

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Einar Hille

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BY RALPH PHILLIPS

EINAR HILLE's many achievements as a mathematician and a teacher made him a major force in the American mathematical community during most of his lifetime. He was at heart a classical analyst, yet his principal work was the creation and development of the abstract theory of semigroups of operators, which culminated in his definitive book on *Functional Analysis and Semi-Groups* (1948, 2). In all, Hille authored or coauthored 175 mathematical papers and twelve books. During the twenty-five years of his tenure at Yale (1938–62), he was the director of graduate studies and as such played an important role in making the Yale Mathematics Department one of the best in the country. He was president of the American Mathematical Society (1947–48), and a member of the National Academy of Sciences, the Royal Academy of Sciences of Stockholm, and the American Academy of Arts and Sciences.

Hille was born in New York City under somewhat unfortunate circumstances in that his parents had separated before his birth and his mother was left with the task of raising him alone. Two years later they moved to Stockholm and remained there for twenty-four years, all that time within a few blocks of a parish church where an uncle of Hille's was then the rector. He wrote of those years: "I was an only child and naturally was spoiled in many ways. That the result did not become completely unfit for human company was largely due to my mother's strong criticism. Nothing was good enough for me, but only my best was good enough for her and as often as not that did not satisfy."

About his early talent, Hille wrote: "My interest in mathematics came fairly late. I recall having had trouble with the seven table in the third grade and that I badly flunked a test on decimal fractions in the sixth grade. But from the ninth grade on mathematics was one of my best subjects and I did outside reading in this subject regularly from the tenth grade on."

Hille entered the University of Stockholm in the fall of 1911 with the aim of becoming a secondary school teacher. For this it was necessary to get a master's degree, and this involved taking three main subjects, at least two of which formed a "teachable" combination. He picked chemistry, mathematics, and physics. Hille started with chemistry but by the end of two years realized that he had little talent in this field. From the beginning, however, to relieve the tedium of the laboratory, Hille started to visit mathematics lectures and thus began his real introduction into the subject under the guidance of Professors I. Bendixson, I. Fredholm, H. von Koch, and docent M. Riesz. Hille received his master's degree in May 1914.

By this time Hille had decided to go for a Ph.D. in mathematics. Riesz suggested the topic of Hille's first mathematical paper, which had to do with properties:

$$L(r) = r\int_0^{2\pi} \left|f'(re^{i\theta})\right| d\theta,$$

where $f(z)$ is holomorphic and $f'(z) \neq 0$ in the disk (:z: <

R). Hille got the licentiate of philosophy degree in 1916 on the basis of this research.

Hille served in the Swedish Army during 1916–17, where he managed to find time to start three investigations, one of which became the basis for his Ph.D. dissertation (1918). This had to do with an integral identity (a form of Green's identity) that he used to obtain information on the distribution of zeros in the complex plane of certain second-order ordinary differential equations. Hille was awarded the Mittag-Leffler Prize for this work. He spent 1919–20 doing office work in the Swedish Civil Service and teaching on the side at the University of Stockholm.

His big break came in 1920 when a fellowship from the Swedish-American Foundation enabled him to spend the year at Harvard University; this was followed by a second year as a Benjamin Pierce Instructor. Encouraged by G. D. Birkhoff, Hille continued to work on the problems engendered in his dissertation (1921, 1; 1922, 1-3; 1923; 1924, 2-6; 1925). Of his later papers, four (1927, 1; 1933, 5; 1943, 1; 1948, 1, in part) are also concerned with the same general set of ideas.

In 1922 Hille moved on to Princeton with the rank of instructor; after his first year he was promoted to assistant professor. To begin with his research continued along the same lines as before; however, he did finish two papers (1924, 1; 1926, 1) at this time on the Dirichlet series, which he had already started in Sweden. Then in 1925 he began working on expansions in terms of Laguerre (1926, 3) and Hermite (1926, 4) polynomials. The latter paper contained results on the Abel summability of such series as well as a study of the Gauss-Weierstrass transform.

With Veblen's help, Hille got a National Research Council fellowship for the year 1926–27 during which he divided his time between Stockholm, Copenhagen, and Göttingen. Three publications grew out of this period (1927, 1, 2; 1928, 1), but they were really incidental to the beginning of a close and fruitful collaboration with J. D. Tamarkin that started around this time and continued up to Tamarkin's untimely death in 1945. A touching account of this collaboration can be found in Hille's "In Retrospect."[1,2]

Hille and Tamarkin started with the problem of the frequency of the characteristic values of linear integral equations (1928, 3; 1931, 1) and continued on to study other phases of integral equations (1930, 2; 1934, 4). This was followed by papers on the Fourier series (1928, 4; 1929, 1; 1930, 3; 1931, 4; 1932, 1; 1933, 1, 2; 1934, 5), Fourier transforms (1933, 3, 6; 1934, 2; 1935, 1, 2) and Hausdorff means (1933, 4, 7, 8; 1934, 3). In all they wrote twenty-six joint papers during the period 1927–37.

Also while at Princeton, Hille became interested in an old problem concerning the width of the strip of uniform nonabsolute convergence of an ordinary Dirichlet series. In 1913 Bohr had shown that this width could be at most one-half and when the summation extends only over the primes that the width is zero. Using a result of Littlewood, Hille was able to prove that, if the summation extends only over those integers that are the product of n primes, the width of the strip is at most $(n-1)/2n$. Finally, F. Bohnenblust, who was Hille's assistant at the time, was able to construct examples for which these upper limits were attained. Two papers (1931, 2, 3; 1932, 3), written jointly with Bohnenblust, contained these results.

In 1933 Hille went to Yale, where he stayed until he reached the mandatory retirement age of sixty-eight in 1962. During all this time his primary responsibility as both teacher and administrator was with the graduate studies. He was director of graduate studies from 1938 to 1962 and as such had a very close relationship with the students. His lectures were always very polished; in fact, he usually wrote them out in longhand in his notebook the night before. In all he had twenty-four Ph.D. students while at Yale. They were Eugene Northrup, 1934; Augustus H. Fox, 1935; William B. Cater, 1939; Irving E. Segal, 1940; Mary K. Peabody, 1944; Joseph S. Leech, 1947; Robert A. Rosenbaum, 1948; Walter J. Klimczak, 1948; Evelyn Boyd (Collins), 1949; Arthur S. Day, 1949; Joseph R. Lee, 1950; John F. L. Peck, 1950; Hai-Tsin Hsu, 1950; Arthur R. Brown, 1952; Thomas L. Saaty, 1952; James L. Howell, 1954; Stephen E. Puckette, 1957; Roger H. Geeslin, 1958; Cassius Ionescu-Tulcea, 1958; Charles A. McCarthy, 1959; Norman S. Rosenfeld, 1959; Saturnino L. Salas, 1959; Thaddeus B. Curtz, 1960; and Sister Mary Zachary Brunell, 1964.

Up to this point in this chronicle my main source has been Hille's own account of his early years published in the essay, "In Retrospect"[3] and preprints entitled "Home, Schools, Avocations" and "Accomplishments." Material for the previous paragraph was taken from Jacobson's article entitled "Einar Hille, His Yale Years, A Personal Recollection."[4] In what follows I have relied on my forward to Hille's selected papers[5] and Yosida's article, "Some Aspects of E. Hille's Contribution to Semi-Group Theory."[6]

In 1936 Hille met Kirsti Ore, the sister of the Yale mathematician Oystein Ore. Einar and Kirsti were married the following year. Kirsti was a devoted wife, and they had two sons, Harald, born in 1939, and Bertil, born in 1940.

Hille's most important research was on the theory of semigroups of operators, which he developed almost singlehandedly over a twelve-year period beginning in 1936. It all started with an investigation on the Gauss-Weiertrass and the Poisson transforms, both of which happened to satisfy the semi-group property: $T(t+s) = T(t)T(s)$. Hille was interested in the degree of approximation to the identity of $T(t)$ for small t, and he found that he could obtain the desired results using only the semi-group property without invoking the explicit form of the transformations (1936, 1).

Hille's next effort in this direction was somewhat tentative. Starting with a semi-group of positive self-adjoint contractions acting on a Hilbert space, he was able to derive a representation theorem and from this prove the analyticity in the semi-group parameter in the right half-plane (1938, 3). At this point he realized the true potential of the theory and he extended the previous results to general holomorphic semi-groups of operators on a Banach space (1938, 4; 1939, 1; 1942, 4). These three papers are the basis for the most beautiful chaper in Hille's book on semi-groups of operators. This chapter contains a basic generation theorem, giving necessary and sufficient conditions for an operator to generate a semi-group holomorphic in a sector, as well as a characterization of the convex hull of the spectrum of the infinitesimal generator in terms of the exponential growth of the semi-group of operators along the various rays in the sector of definition.

Hille spent 1941–42 on sabbatical at Stanford, where he became involved with G. Polya, A. C. Schaeffer, and G. Szego in the extension of certain oscillation theorems of Polya and Wiener on Fourier series to classical orthogonal polynomials (1942, 1, 2; 1943, 1). In the spring of 1942 he joined forces with Max Zorn, and together they wrote a paper on additive semi-groups of complex numbers (1943, 2).

In 1942 (starting with the paper [1942, 3]), Hille began an attack on the larger class of semi-groups of operators that are merely strongly continuous. In this paper he discovered the representation of the resolvent of the generator in terms of the Laplace transform of the semi-group. It should be noted that Hille was mainly motivated in this by the theory of analytic functions of exponential type as developed by Polya and not, as one might expect, by the standard Laplace transform approach to the initial value problem in partial differential equations. In fact, it was not until 1949, with the research of Yosida[7] on the diffusion equation, that this most important application of the theory of semi-groups of operators was considered.

In August of 1944 Hille delivered the colloquium lectures at the American Mathematical Society meeting and immediately thereafter he started in earnest writing his book,[8] *Functional Analysis and Semi-Groups*, which was finally published in 1948. It was both a textbook on functional analysis and a monograph on the theory of semi-groups of operators. As far as I know, this was the first time that functional analysis was presented as a tool for classical analysis. In addition to the usual theory of Banach spaces and linear transformations, Hille was able to organize into a unified whole the calculus of vector-valued functions, function theory for vector-valued functions, and the operational calculus. It should be noted that this book was for many years one of the principal texts on functional analysis.

The basic result, giving necessary and sufficient conditions for a closed linear operator A with dense domain to be the infinitesimal generator of a strongly continuous semigroup of contraction operators, appeared in print for the first time simultaneously in Hille's book and in a paper by Yosida.[9] This result is now referred to as the Hille-Yosida generation theorem. The logarithm of the norm of a semi-group of operators is a real-valued subadditive function on the half-line. With this in mind, Hille extended the earlier theory on subadditive functions on the positive integers to measurable subadditive functions on the half-line.

Hille's chapter on ergodic theory is based on an earlier paper (1945, 1) and deals with the Abel limit:

$$(A) - \lim_{\lambda \to 0,\infty} T(t) = \lim_{\lambda \to \infty,0} \lambda R(\lambda),$$

where $R(\lambda)$ is the Laplace transform

$$R(\lambda) = \int_0^\infty e^{-\lambda t} T(t)\, dt.$$

By means of Tauberian theorems, Hille was able to show that under certain auxiliary conditions Abel summability implies the stronger Cesaro summability. He also proved that the Abel limits at both 0 and ∞ were projection operators related to the semi-group.

The spectral theory chapter is outstanding. It is based on an operational calculus explicitly constructed for the generators of semi-groups of operators. The basic relation between the function $f(\lambda)$ and the corresponding operator $f(A)$ is given by

$$f(\lambda) = \int_0^\infty e^{\lambda t}\, d\beta$$

and

$$f(A) = \int_0^\infty T(t)\, d\beta,$$

A being the infinitesimal generator of $T(t)$; thus, both $f(\lambda)$ and $f(A)$ are Laplace-Stieltjes transforms. In earlier versions of the operational calculus, $f(\lambda)$ was always taken to be holomorphic on the spectrum of A. However, Hille treated functions $f(\lambda)$, which are holomorphic in the interior of the spectrum of A but may be merely continuous on those points of the spectrum that lie on the abscissa of convergence for the Laplace-Stieltjes transform of β. A spectral mapping theorem relates the fine structure of the spectrum of A to that of $T(t)$ by means of an ingenious argument (1942, 4). It should be remarked that Hille made no use of the theory of Banach algebras in this discussion even though his book contains an appendix on Banach algebras.

Part three of Hille's book deals with a wide variety of examples of semi-groups taken from classical analysis. The chapters on trigonometric semi-groups and translation semi-groups are related to factor sequences for Fourier series and factor functions for L_p spaces, which Hille dealt with in earlier papers ([1924, 1; 1926, 4; 1933, 1, 5]). The chapter on partial differential equations is somewhat disappointing in that it did not anticipate the most successful approach to the subject via the Hille-Yosida theorem. The concluding chapter contains a rich variety of examples taken from summability theory, Markoff chains, stochastic processes, and fractional integration. In each of these examples Hille threw new light on the subject with his semi-group theory.

In 1948 Hille gave his retiring presidential address on the theory of Lie semi-groups of operators (1950, 1) at the annual meeting of the American Mathematical Society. This work contains both a study of the underlying parameter group π and an investigation of its representation $(T(p); p \in \pi)$ as bounded linear operators on a Banach space. Hille showed that corresponding to every canonical sub-semi-group there is an infinitesimal generator, that these generators form a positive cone, and that they satisfy the analogs of the three fundamental theorems of Lie.

In 1952 Hille asked me to collaborate with him on the second edition of his book.[8] I was occupied with this task during much of the 1952–53 academic year and all of 1953–54, which I spent at Yale. I found Hille to be a generous colleague, extremely patient, and perhaps a little too permissive for the good of the book. Although the new edition (1957, 2) is one-and-a-half times the size of the original, it is largely in fact and entirely in spirit very much like the original edition.

The 1957 edition consists of a thorough reworking of the first edition plus several new results necessary to bring it up to date. The theory of commutative Banach algebras is introduced early in the book and plays a major role in the chapters on spectral theory and holomorphic semi-groups. The influence of Yosida and, to some extent, Feller is quite evident; and of course I took advantage of my being coauthor by including my own results on extended classes of semi-groups (distinguished by their behavior at the origin) and their generating theorems, perturbation theory, the adjoint semi-group, the operational calculus, and spectral theory. There is also a new chapter on Hille's theory of Lie semi-groups of operators and an expanded section on the integration of the Kolmogoroff differential equations based mainly on two of Hille's papers (1954, 5, 6). The chapter on partial differential equations was omitted because by that time it required a book of its own. However, we did include a discussion of the abstract Cauchy problem that had been initiated by Hille and on which we had both worked.

Yosida's 1949 paper[10] on the diffusion equation showed that semi-group theory was an ideal tool for studying the initial value problems in mathematical physics. It was typical of Hille that he launched into an attack on this problem on two different levels: the abstract and the concrete. On the abstract level he formulated what he called the abstract Cauchy problem (ACP): for a given Banach space

X and y_0 in X and linear operator U with domain $D(U)$, find a solution to the problem $y'(t) = U[y(t)]$, $t > 0$ such that $\lim_{t \to 0} y(t) = y_0$ and $y(t)$ is absolutely continuous and $y'(t)$ exists and belongs to $D(U)$ for all $t > 0$. A solution is said to be of normal type if as $t \to \infty \lim\sup t^{-1} \log |y(t)| < \infty$.

Hille investigated the ACP in a series of papers (1951, 2; 1953, 3; 1954, 1, 4; 1957, 1). He showed that the ACP has at most one solution of normal type if U is a closed operator whose point spectrum is not dense in the right half-plane. On the other hand, if the spectrum of U covers the entire plane and X is an L-space, the ACP may have explosive solutions (1950, 3; 1954, 1). It is obvious that if U generates a semi-group of operators, then the ACP is solvable for every y_0 in $D(U)$, and the solutions will be of uniform normal type for all y_0 of norm ≤ 1. Hille was able to prove the converse assertion (1954, 1); after reading his manuscript, I managed to prove a stronger form of the converse theorem, and this is what appears in our book.

On the concrete level, Hille began working on the forward diffusion equation:

$$\frac{\partial f}{\partial t} = Lf \equiv \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(b(x)f) - a(x)f\right\} \quad \alpha < x < \beta,$$

acting on $L_1(\alpha, \beta)$. He was able to obtain new proofs (1949, 1) of Yosida's results under less restrictive conditions. These results were communicated to Feller, who suggested to Hille that he attack the backward diffusion equation:

$$\frac{\partial g}{\partial t} = Cg \equiv b(x)\frac{\partial^2}{\partial x^2}g + a(x)g \quad \alpha < x < \beta,$$

acting on $C[\alpha, \beta]$, by the same methods. This suggestion proved very fruitful. Another effect of this interchange of ideas was to get Feller interested in the problem, and with the aid of Hille's preliminary results, Feller[11] was able to find a fairly complete solution for both the forward and the backward equations.

Hille's investigation proceeded more slowly, and it was not until 1954 that he published a comprehensive account (1954, 1); see also (1956, 2). He had given himself the problem of finding necessary and sufficient conditions on the coefficients a and b that the maximally defined operators L and C (i.e., with no boundary conditions) generate semi-groups of positive contraction operators. Although superseded in many ways by Feller, Hille's paper contains a wealth of additional information about L and C.

In 1953 Hille began working on the Kolmogoroff differential equations:

$$Y'(t) = AY(t) \quad \text{and} \quad Z'(t) = Z(t)A,$$

where the matrix $A = (a_{1j})$ is a Kolmogoroff matrix:

$$a_{ii} \leq 0, \; a_{ij} \geq 0 \;\text{ for } i \neq j$$

and

$$\Sigma_i \, a_{ij} = 0 \;\text{ for all } i,$$

and both Y and Z belong to the Markoff algebra M of matrices $U = (u_{ij})$, $u_{ij} \in C$, such that

$$|U| = \sup_i \Sigma_j \, 1 u_{ij} 1 < \infty.$$

The domain of A is defined as $D(A) = (U \in M; AU \in \mathrm{M})$, and the subspace M_A is the closure of $D(A)$.

One of Hille's main results (1953, 5; 1954, 3) is the following: Suppose A is triangular, that is, $a_{ij} = 0$ for $j > i$, and define the restriction A_0 of A with domain

$$D\,(A_0) = (U \in D(A); AU \in M_A).$$

Then A_0 generates a strongly continuous semi-group of transition operators satisfying both Kolmogoroff differential equations. Another result (1954, 6) has to do with null solutions: suppose that $Z(t)$, satisfying the second Kolmogoroff differential equation, is differentiable in the norm topology for $t > 0$, that $\lim_{t \to 0} \|Z(t)\| = 0$ and that $Z(t)$ is of normal type. Then for A triangular and arbitrary B in M, the equation $Q'(t) = Q(t)(A + B)$ has a solution with the same properties.

Hille followed this up with a series of papers (1961, 2; 1968, 2; 1966, 2) on differential equations in a Banach algebra of the form

$$w'(z) = f(z)\, w(z).$$

Here z is a complex variable while $f(z)$ and $w(z)$ belong to a complex noncommutative Banach algebra B with unit $f(z)$ being analytic in z. He studied the solutions of this equation in a simply connected domain of holomorphism of $f(z)$, in a neighborhood of a simple pole of $f(z)$, and in a partial neighborhood of a multiple pole.

Hille also became interested in transfinite diameters at about the same time, that is, 1961, and pursued this off and on for another five years. To understand the problem, we go back to Kolmogoroff's abstract definition of an averaging process A:

(i) A assigns to any finite set of positive numbers $(x_1,\dots,x_m)$ positive average $A(x_1,\dots,x_m)$;

(ii) $A(x_1,\dots,x_m)$ is continuous, symmetric, and strictly increasing in each argument;

(iii) $A(x,\dots,x) = x$;

(iv) If $A(x_1,\dots,x_k) = y$, then

$$A(x_1,\dots x_k, x_{k+1},\dots,x_m) = A(y,\dots,y,x_{k+1},\dots,x_m).$$

Given a compact metric space E, consider sets of n points, and for each set define the x_i's to be the $m = n(n-1)/2$ distances between them. Then, the $A(x_1,\ldots,x_m)$ are bounded as a function of these sets with a maximum value $\delta_n(E)$. It can be shown that $\delta_{n+1}(E) \leq \delta_n(E)$. The transfinite diameter is defined as

$$\delta_0(E) = \lim_{m\to\infty} \delta_m(E).$$

In 1923 Fekete proved that for a compact set E in the plane the transfinite diameter was equal to the Chebyshev constant $\chi(E)$. It is possible to extend the notion of the Chebyshev constant to compact sets E in a metric space. Hille proved (1962, 1) that in this generality $\chi(E) \geq \delta_0(E)$. In two further papers on this subject (1965, 1; 1966, 1), he calculated the transfinite diameters of the unit spheres of some complex Banach spaces.

After retiring from Yale in 1962, Hille stayed in New Haven for two more years and then started on a nomadic existence that took him from one visiting teaching post to another for the next eight years before ending up at the University of California at San Diego. In between he worked at the University of California at Irvine, the University of Oregon in Eugene, and the University of New Mexico in Albuquerque.

During all this time he continued to be very active mathematically. Starting in 1959, he produced nine textbooks and, in addition, investigated the Thomas-Fermi equation (1969, 2; 1970, 1), Emden's equation (1970(2), 1972(5,6)), and the Briot-Boquet equations (1978, 1–4).

The last time I saw Einar was at the Laguna Beach Conference in his honor (January 8–9, 1980). At the time he was terminally ill, but he somehow managed to take leave of the hospital and attend the conference under Kirsti's gentle care. It was typical of Hille that even in his weakened condition he was able to deliver an interesting lecture—his last.

The effort that Hille put into mathematics, in all its aspects, was awesome. I suspect that deep in his subconscious was always the quote from Kipling that he inserted as the frontispiece to the first edition of *Functional Analysis and Semi-Groups*:

"And each man hears as the twilight nears,
to the beat of his dying heart,
The devil drum on the darkened pane:
"You did it, but was it Art?"

There is no doubt in my mind but that Hille carried the right to be satisfied on this score.

NOTES

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2. Einar Hille, *Functional Analysis and Semi-Groups, Am. Math Soc. Colloq. Publ.*, XXXI (1948), xii + 528.
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