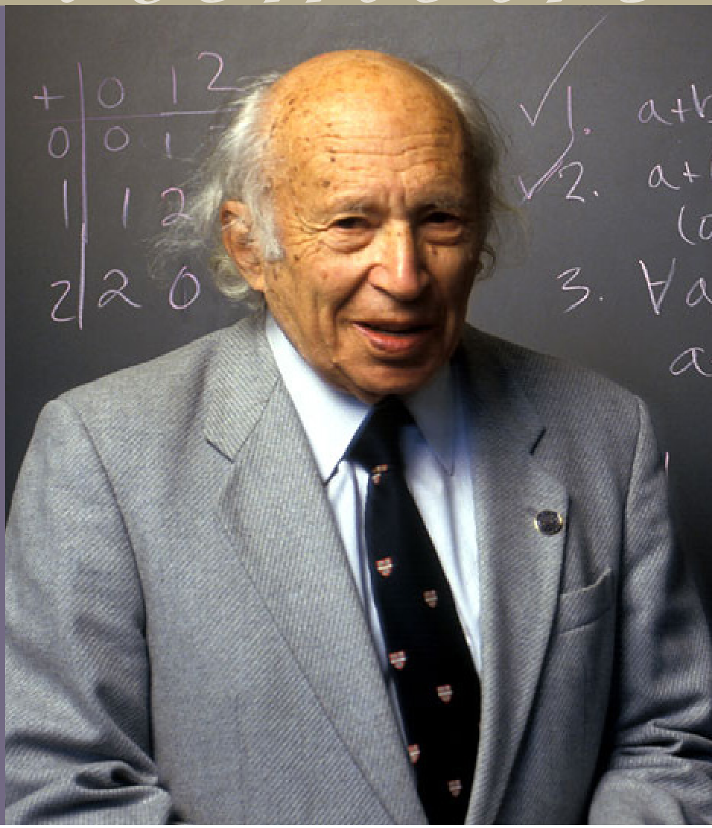
Israel M. Gelfand

1913-2009

BIOGRAPHICAL *Memoirs*



A Biographical Memoir by
A. A. Kirillov

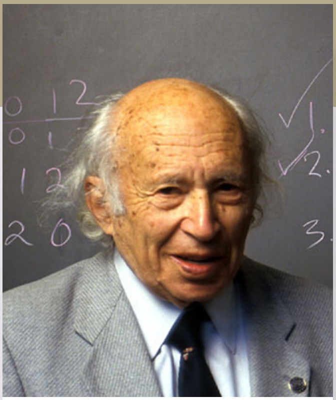
NATIONAL ACADEMY OF SCIENCES

ISRAEL MOISEEVICH **GELFAND**

September 2, 1913 – October 5, 2009

Elected to the NAS, 1970

Israel Moiseevich Gelfand was one of the most influential mathematicians of the twentieth century. He was the author of more than 800 articles and thirty books in almost all areas of mathematics and in theoretical biology. It is impossible to describe his long and controversial life in a short tribute. Besides, I was witness to only a part of the numerous Gelfand activities. In particular, I know very little about his works in applied mathematics during the 1940s and 1950s and about his biological seminars in Moscow from the 1960s to the 1980s and at Rutgers University from 1990 on.



Photograph courtesy of Rutgers University.

By A. A. Kirillov

Israel M. Gelfand entered my life as my university teacher, as the head of the legendary Gelfand's Seminar, as the founder and the chief editor of the journal *Functional Analysis and Its Applications*, as the president of Moscow Mathematical Society, and as an organizer of the Mathematics Correspondence School for high school students.

I think that the best way to give readers a genuine idea about Gelfand's personality is to present some facts of his quite unusual life and quote several statements by I. M. himself and by his students, colleagues, rivals, friends, and other contemporaries. I shall speak only about what I remember myself, or what was published or said openly to a wide audience.

Gelfand was born on September 2, 1913, in the small town of Okny in the Kherson Governorate of the Russian Empire (now the Odessa region of Ukraine). Several different states and political regimes replaced one another in governing this place between 1917 and 1923. Life was not easy even for relatively happy people.

At 15, he was hospitalized for appendicitis. On his way there, he asked his parents to buy him a calculus text. In those days, appendectomies required a twelve-day stay. The bedridden boy entertained himself by mastering the materials. Later, he used to say that his knowledge of analysis remains incomplete, because he only had the first part of the textbook.

As a ninth grader, Gelfand was expelled from a technical school because his father, who operated a mill and had an assistant, was designated a capitalist and denied the right to vote. Gelfand then went to Moscow in 1930, before completing his secondary education. There he took on a variety of different jobs (including doorkeeper at Lenin's library), but he also began to teach mathematics. At that time in Moscow, there were many different institutes where mathematics was taught in evening classes, and Gelfand taught elementary mathematics in several of these institutes. A little later, he progressed to teach more advanced mathematics. In addition to his teaching, he also visited lectures at Moscow State University. The first course he attended was the theory of functions of a complex variable taught by Mikhail Alekseevich Lavrentyev.

Two years later, Gelfand was admitted as a graduate student to Moscow State University under the direction of Andrei Nikolaevich Kolmogorov. He was fortunate to be in a strong school of functional analysis where he received much support from other mathematicians, such as Abraham I. Plessner and Lazar A. Lusternik. Gelfand received his Ph.D. in 1935, with a thesis entitled "Abstract functions and linear operators." In five years, he earned a doctor of science degree. (This degree has no U.S. equivalent and is similar to D.Sc. in the Commonwealth.)

Early Career

Beginning in 1939, he worked at Moscow State University as an associate professor and in 1941 became a full professor. He also worked at the Keldysh Institute of Applied Mathematics (KIAM), which was at that time a non-declared division of the Steklov Institute of Mathematics. In 1951 and 1953, he was awarded the Stalin Prize for works in applied mathematics and was elected as a corresponding member of the Academy of Science.

Photo courtesy of the Gelfand family archives.

But his career was not smooth. After several years at Moscow State University, he was denied the right to give regular lectures for students, so, he concentrated on his seminar. Although repeatedly invited to International Congresses, Gelfand did not attend any Congresses held abroad until the 1986 Congress in Berkeley. The president of the Soviet Academy of Sciences from 1961-1975, mathematician Mstislav Keldysh, reportedly commented: "The harm from not letting Gelfand go abroad has already exceeded any potential harm from letting him go."

In 1943, Gelfand established his legendary seminar, which operated independently of the university and was open to everyone. In my time at the university (1957-87), the seminar was held every Monday and was the most popular meeting place for many mathematicians, not only from Moscow. Gelfand was such a popular lecturer that he spoke far more than any other speaker. Indeed, the Independent Moscow University recently published on its website the notes of the seminar talks from 1964-84, prepared by M. I. Shubin. In the notes, Gelfand is listed as having lectured 106 times. The next most-frequent speaker, A. A. Kirillov, comes in at only thirty-three times!

Shift to Biology

From 1958 onwards, Gelfand became interested in problems in the fields of biology and medicine. In 1960, together with S. V. Fomin and other scientists, he set up the Institute of Biological Physics within the USSR Academy of Sciences. In particular he became interested in cell biology and also in experimental work in addition to the theoretical work that was his first interest. One of the reasons for his interest in biology was personal tragedy: his third son, Sasha, suffered from leukemia and died when he was six. In 1960, the last winter of Sasha's life, the family lived at their dacha (summer cottage) near Moscow, where the old nurse looked after him. At that time I was just married and lived with my wife, my parents, and my sister in a small, one bedroom apartment. Gelfand offered me and my wife, Louiza, an attic room in his dacha, and for two months it was our first family home.

Creation of the Correspondence School

The years 1960-1990 were a time of tremendous activity for Gelfand. His seminar became known worldwide. Almost all the remarkable mathematicians from Moscow, Leningrad, Kiev, as well as most foreign visitors, attended it at least once. The audience included mathematicians of all ages and status. (Gelfand used to say that his seminar is for students, good graduates, and excellent professors.)

In 1963, the rector of Moscow State University, I. G. Petrovskii, tried to convince Gelfand to join A. N. Kolmogorov in the organization of a mathematical boarding school. After some reflection, Gelfand refused this idea and proposed instead to create a correspondence school in mathematics, which would be popular and could attract much more talented young people than an elitist boarding school. Both projects were successfully realized. The Kolmogorov Boarding School started in 1963, the Gelfand Correspondence School a year later. The number of students during first five years expanded from 6,000 to 16,000. Gelfand himself, together with his colleagues and students, wrote introductory textbooks and instructed university students in how to work with schoolchildren.

In 1966, Gelfand succeeded Kolmogorov in the position of president of the Moscow Mathematical Society, a post he remained in until 1970. This period would prove to be the flowering time of the society. Neither before, nor after, was there such fireworks of remarkable talks in an over-crowded room 16-24 (big auditorium for 200 seats).

In 1967, Gelfand founded *The Journal of Functional Analysis and Its Applications* and ran it for twenty-three years. He succeeded in attracting to the editorial board many actively working mathematicians, not only experts in functional analysis. He was a very energetic and rather meticulous chief editor. The meetings of the board, which he always ran in person, often went on several hours. And I will discuss this topic more later.

The Cold War Era

The end of 1960s was also the end of Cold War thaw, which had started in 1954. The general situation was described by Roland L. Dobrushin thus: “the Soviet Power came to mathematics.” The formal reason for the new policy was the so-called “Letter by 99,” signed by many well-known and young mathematicians in protest against the psychiatric confinement of logician Alexander Essenin-Volpin.

The dean of the mathematical department, N. V. Efimov, was dismissed and replaced by Soviet Party secretary Petr M. Ogibalov. All the activity of the department was subordinated to the Party Committee. Not only hiring (and firing), membership in editorial boards, and trips abroad, but even the distribution of teaching duties or the subjects of special courses and seminars had to be approved by party authorities. Special attention was paid to the entrance exams. Only a rather narrow circle of trusted people was admitted to proctor them. The situation worsened with the death of two remarkable rectors: Ivan G. Petrovskij (1971) and Rem V. Khokhlov (1977).

This complicated and controversial period in Russian history is marked by three slogans: acceleration, glasnost (publicity), and perestroika. For the Russian scientific community, it had two opposite effects: a relative liberation (the cancellation of censorship, the possibility of travel abroad) and an essential decline of living standards and professional prestige. Science as a profession became much less attractive, and many talented young people chose other occupations. It also led to a massive brain drain. Moscow State University lost its status of the best mathematical center in the world, which it had in the Golden Era of the 1950s and 1960s owing to the unprecedented concentration of first-class mathematicians in one place.

Gelfand at Rutgers

In 1990, Gelfand made his first long visit abroad. He lectured at Harvard University, Yale University, the University of Pennsylvania, Princeton University, and finally Rutgers University, where he remained as a faculty member. The same year, he established the Gelfand Correspondence Program in Mathematics (GCPM), an analog of the school he organized in Russia. Though the scale and the effect of this endeavor was much smaller than in Russia, I think that the impact of it for mathematical education in the United States is comparable with that of many-million-dollar investments made by the government. His principle was: “In our century of rapid changes, it is impossible to know everything. The goal is to learn how to learn.”

He also continued his very active scientific work in mathematics and biology. The number of his publications in the last two decades of his life reached over one hundred. He also continued to run two seminars (mathematical and biological) at two Rutgers campuses.

Contributions to Mathematics

Gelfand was one of the very few mathematicians who understood almost all domains of mathematics and successfully worked in many of them. I shall try to describe the most brilliant of his results.

Functional Analysis

The Moscow mathematical school was traditionally strong in function theory. In the 1930s, mathematicians in Poland and France brought a new breath of discovery: functional analysis. This new domain expanded quickly and attracted many young mathematicians in Russia. Through the initiative of Kolmogorov, the course of functional analysis (under the name “Analysis-3”) was introduced in the mathematical curriculum at Moscow State University. It is not surprising that Gelfand’s first mathematical results were related to functional analysis. In his series of papers written between 1935 and 1940, beginning with his dissertations, Gelfand introduced several basic constructions and obtained a lot of remarkable results. Using his lemma of boundedness of convex functionals, he gave the uniform proof of many earlier results by Radon, Orlich, and

Danford, describing the general form of continuous and compact operators between classical Banach spaces:

$$c_0, c, l^1, l^\infty, C[a, b], L^1[a, b], L^\infty[a, b], V[a, b]$$

He extended the famous Stone's formula for one-parameter groups of unitary operators in Hilbert space

$$U(t) = \exp(tA)$$

to the groups of bounded operators in Banach spaces. In the same paper, he proposed the method of smoothing operators. Later, Lars Gårding noted that this method works in more general situations, when an arbitrary Lie group G acts by linear operators on a Banach space V. The point is that V always contains a dense subspace V^∞, now called Gelfand- Gårding space, where the action is differentiable. It allows the use of representation theory of Lie groups, the machinery of Lie algebras, and their enveloping algebras.

The important next step was the passage from Banach spaces to Banach algebras. Gelfand introduced and thoroughly investigated the important class of complex Banach algebras with involution (now they are known as C^*-algebras).

He discovered that every commutative C^*-algebra with unit is isomorphic to the algebra $C(X)$ of continuous functions on a compact topological space X. The points of X correspond to the maximal ideals of A. In a joint paper with Mark A. Naimark, he showed that all non-commutative C^*-algebras can be realized as uniformly closed algebras of bounded operators on a Hilbert space. This fundamental result was obtained in a joint paper with Dmitri Raikov. Namely, it was shown that every locally compact topological group has a complete system of unitary irreducible representations.

In Gelfand's publications, both his results and his methods of proof were new and contained several useful constructions: maximal ideals, smoothing operators, the Gelfand, Naimark, and Segal construction, and new applications of the Krein-Milman theorem. All these results made Gelfand a world-renowned leader in functional analysis.

Jean Dieudonné, in his *History of Functional Analysis*, defined this domain as "the study of topological vector spaces and mappings $u: \Omega \to F$ from a subset Ω of a topological vector space E into a topological vector space F, these mappings being assumed to satisfy various algebraic and topological conditions." In Moscow, however, the following "constructive definition" of functional analysis was popular: FA is the part of mathematics developed by I. M. Gelfand with his students and colleagues, discussed in his seminar, and published in his journal.

You can judge Gelfand's own opinion of functional analysis from the following story. Once, being the youngest member of the editorial board of the *Journal of Functional Analysis and Its Applications,* I shared with I. M. my doubt that a certain paper would fit the subject of the journal. He asked, "Is it a good paper? "I replied, "Yes!" To which he said, "A good article always fits the subject."

Differential Operators

I mention here only three results obtained by Gelfand in connection with his applied studies. The first deals with the explicit expansion of an arbitrary function by eigenfunctions of a given differential operator. As a byproduct, Gelfand obtained a beautiful formula, establishing an isomorphism between the Schwartz space $\boldsymbol{S}(\mathbb{R}^n)$ and the $\Gamma(E, \mathbb{T}^n)$ space of smooth sections of the universal line bundle over the *2n*-dimensional torus. For n=1 the formula is

$$f(x) \rightsquigarrow \varphi(t,\tau) = \sum_{k\in\mathbb{Z}\rtimes\ltimes} e^{-ik\tau t} f(t+k), \quad \varphi(t,\tau) \rightsquigarrow f(x) = \frac{1}{2\pi}\int_0^1 \varphi(x,\tau)d\tau.$$

I think it is the simplest way to derive the Fourier transform on $\mathbb{R}$ from the Fourier transform on $\mathbb{T}$.

The second is the famous formula developed by Gelfand and Levitan allows us to determine the potential q of the Sturm-Liouville operator on $(0, \infty)$

$$L(y) = y' - q(\chi)\, y = 0 \quad \text{with boundary conditions } y(0) = 0,\ y'(0) = h$$

knowing the spectral data of L. More precisely, let $\varphi(\chi, \lambda)$ be the unique eigenfunction of L with the given boundary conditions and the real eigen-value λ.
For any $f \in L^2(0, \infty)$ put $E_f(\lambda) = \int_0^\infty f(x)\varphi(x, \lambda)dx.$ It is known, that there exists a unique Borel measure $d\rho$ on $\mathbb{R}$ (the spectral measure), given by a weakly increasing function $\rho(\lambda)$ such that

$$\int_0^1 f^2(x)dx = \int_{-\infty}^{\infty} E_f^2(\lambda)d\rho(\lambda).$$

In the Gelfand-Levitan paper of 1951 the explicit formula was found, which expressed $q(\chi)$ in terms of $\rho(\lambda)$. It was soon applied by Kruskal et al. to solve integrable systems such as the Korteweg-De Vries equation, by applying the scattering and inverse scattering transforms associated with the Lax operator. An excellent review of this result was recently done by the Fields medalist Terence Tao.

The third result is the formulation of the index problem for elliptic differential operators posed by Gelfand in 1960. Thinking about differential operators with partial derivatives, Gelfand predicted that the index of elliptic operator can be expressed in terms of topological properties of its symbol. This problem was solved in a few years in the famous series of papers by Atiyah-Singer.

Generalized Functions and Distributions

The famous delta function was introduced by Paul Dirac in 1930 as a "convenient notation" for some computations in quantum theory. The rigorous mathematical definition of generalized derivatives was introduced by S. L. Sobolev in 1935. But only after L. Schwartz's book *Théorie des Distributions* in 1950-51 did the notion became widely known and used.

Gelfand was the first to use generalized functions in representation theory. One of his beautiful results is the thorough study of homogeneous distributions in $\mathbb{R}^n$ and the description of Lorentz-invariant differential equations.

Another important result is the definition of generalized characters for infinite-dimensional representations of Lie groups and the discovery that for complex classical Lie groups these generalized functions are regular, even analytic in an open dense domain. This result was extended by Harish-Chandra to all reductive Lie groups and is considered the most important achievement in non-commutative Harmonic analysis.

Representation Theory

In 1937 the Kharkov Mathematical Society published book containing Russian translations of nine papers by G. Frobenius about group representations. Gelfand told me that he read this book very attentively and thought a lot about the subject. The result of these thoughts was the understanding that in the theory of infinite-dimensional representations theory of classical Lie groups the crucial role is played by commutative Banach algebras–Gelfand's favorite subject in the 1930 and 1940s.

Gelfand's main idea was that the infinite-dimensional unitary representations of the classical Lie groups admit as explicit and beautiful descriptions, as do finite-dimensional representations. Moreover, in a certain sense, the infinite-dimensional representations are simpler: the finite-dimensional representations are "singular points" in the variety of all representations. The Banach algebra approach resulted in Gelfand's famous series of papers with Mark A. Naimark about representations of complex classical Lie groups and with Mark I. Graev about representations of real classical Lie groups.

The next idea, originating in algebraic geometry, consists in viewing the underlying field as a parameter of the theory. In this way, representations of complex, real, *p*-adic, adelic, and finite groups become a part of a more general scheme. In his talk, presented to the International Congress of Mathematics in Stockholm in 1962, I. M. Gelfand considered possible extension of the orisphere method to the algebraic groups over *p*-adic fields and over the ring of adeles.

At the same time, a similar idea was proposed by Robert Langlands. Now it is known as "Geometric Langlands Program." Among the main contributors to this program are many of Gelfland's students and members of his seminar, including Alexander Beilinson, Alexander Braverman, Edward Frenkel, Dennis Gaitsgory, Victor Ginzburg, and Vladimir Drinfeld. Recently Anton Kapustin and Edward Witten described a connection between the geometric Langlands correspondence and S-duality, a property of certain quantum field theories.

Another aspect of the algebraic-geometrical approach to representation theory was developed in our joint paper on the "birational classification" of Lie algebras (IHES, No. 31). Gelfand's original idea was that enveloping algebras of Lie algebras are the true object of study in non-commutative algebraic geometry. The fecundity of this approach has now been substantiated by numerous papers. The theory of D-modules and its applications to representation theory, now very popular, appeared under the influence of these papers and their discussions at the Gelfand seminar.

I want to finish this section with a discussion of the so-called Gelfland-Tsetlin patterns, introduced in two short papers in *Doklady Notes* in 1950. These notes became famous immediately after publication and remain the most quoted by mathematicians and physicists. It is notable that these papers were written for physicists in a manner that is popular in physics. In particular, there are no proofs: the explicit formulas for irreducible representations is all that physicists need. I also close with this discussion because I myself made a small contribution to this subject (see the special issue of *J. Geom. Phys.* 1988, 5(3):473-482, devoted to the 75th anniversary of Gelfand).

The papers in question give the bases for all irreducible finite-dimensional representations of classical groups in a unified form. Élie Cartan and Hermann Weyl already knew that irreducible representations V of reductive groups are enumerated by so-called highest weights. For $GL(n)$ they are monotone strings of integers $\lambda = (\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n)$. Gelfand and Tsetlin showed that the basic vectors in V can be naturally labeled by triangular tables of the form

$$
\begin{array}{ccccccccccc}
m_{1,1} & & m_{1,2} & & \cdots & & \cdots & & m_{1,n-1} & & m_{1,n} \\
& m_{2,1} & & m_{2,2} & & \cdots & & \cdots & & m_{2,n-1} & \\
& & m_{3,1} & & \cdots & & \cdots & & m_{3,n-2} & & \\
& & & \cdots & & \cdots & & \cdots & & & \\
& & & & m_{n-1,1} & & m_{n-1,2} & & & & \\
& & & & & m_{n,1} & & & & &
\end{array}
$$

where $m_{1,k} = \lambda_k$ and all other entries satisfy the "betweenness" property: $m_{k,l} \le m_{k+1,l} \le m_{k,l+1}$. The explicit formulas for the action of generators of the Lie algebra $\mathfrak{gl}(n)$ in this basis were also given. Moreover, these bases are well adapted to restriction to a subgroup. Namely, the irreducible space V_λ is reducible with respect to the action of $GL(n-1)$. To decompose it to irreducible parts $V\mu$ with the highest weight μ, it is enough to erase the first row λ and collect together the tables with the second row μ.

Of course, the real reason for this phenomena is the remarkably simple form of the "branching rule" for some of subgroups H of a compact group G.

Namely, let us call a subgroup $H \subset G$ a "big subgroup" if the following equivalent properties take place:

(*) For any unirrep π of G, the restriction $\mathrm{Res}_H^G \pi$ has a simple spectrum.

(**) For any unirrep ρ of H, the unirrep $\mathrm{Ind}_H^G \rho$ has a simple spectrum.

(***) The *relative group algebra* $\mathrm{Fun}(G)^H$ is commutative.

Actually, the notion of a big subgroup does not occur in Gelfand's papers. Instead, he introduced what is now called the *Gelfand pair*. It is a subgroup $H \subset G$ with slightly weaker properties:

$(*)'$ For any unirrep π of G, the restriction Res $^G_H\pi$ contains the trivial representation 1 exactly once.

$(**)'$ For a trivial unirrep 1 of H, the induced unirrep Ind $^G_H 1$ has a simple spectrum.

$(***)'$ The algebra Fun($H\backslash G/H$) is commutative.

My old dream was to find a non-commutative analog of the Poisson summation formula. For this one needs a discrete set X_n, dual to the compact group U(n). For a compact Lie group G, the space Fun(G) of functions on G has the natural basis formed by the matrix elements $\pi_{i,j}(g) = (\pi(g)v_i, v_j)$ of unirreps $\pi(g)$. In the case $G = U(n)$, we need two Gelfand-Tsetlin patterns with the same upper row to label a matrix element. It is convenient to glue together these triangular patterns along the common upper side. In this way, we get a square matrix with integer entries, whose rows and columns are (weakly) monotone.

For instance, the matrix elements of the unirrep $\pi_{m1,m2}$ of $U(2)$ are enumerated by matrices

$$M = \begin{pmatrix} m_1 & p \\ q & m_2 \end{pmatrix} \text{ with } m_1 \leq p, q \leq m_2.$$

There are $(m_2 - m_1)^2$ such matrices in accordance with dim $\pi_{m1,m2} = m_2 - m_1$. The collection X_n of $(n \times n)$-matrices with integer entries, which are weakly monotone in rows and column, is a beautiful candidate for the role of dual object to the compact group $U(n)$.

Gelfand and Tsetlin considered only unitary and orthogonal groups. I found that the analogous patterns and bases exist also for the series of quaternionic groups $G_n = U(n,\mathbb{R})$, the maximal compact subgroups in Sp($2n$, $\mathbb{C}$). The highest weight for G_n looks like

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0).$$

The branching rule is more complicated, since G_{n-1} is no longer a big subgroup in G_n. The spectrum of $Res\ ^{G_n}_{G_n-1}\pi_\lambda$ consists of those ρ_μ, for which $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq 0)$ satisfied the condition: there exists an intermediate weight $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0)$ such that the table

$$\begin{array}{ccccccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_n & & 0 \\ & \nu_1 & & \nu_2 & & & \cdots & & & \nu_n & \\ & & \mu_1 & & \mu_2 & & \cdots & & \mu_{n-1} & & 0 \end{array}$$

satisfies the betweenness condition. Moreover, the multiplicity of ρ_μ is equal to the number of intermediate weights ν. It suggests the enumeration of matrix elements by monotone antisymmetric $2n + 1 \times 2n + 1$-matrices with integer entries.

$$\text{For } n = 1 \text{ they look like } M = \begin{pmatrix} m & p & 0 \\ q & 0 & -p \\ 0 & -q & -m \end{pmatrix}.$$

A challenging problem is to interpret it as a co-inclusion of $U(n,\mathbb{H})$ into $U(2n + 1,\mathbb{C})$.

Representation of Some Infinite-Dimensional Lie Groups

Some infinite-dimensional groups arise as the symmetry groups of mathematical models of physical systems. They are infinite-dimensional Lie groups that possess well-defined Lie algebras. Mathematically speaking, they are symmetry groups of vector bundles E with a fiber F over a smooth manifold M. Often such a group is an extension of the group $Diff M$ of diffeomorphisms of a smooth manifold M by a so-called current group of transformations, preserving fibers. (Locally they are matrix-valued functions on M with point-wise multiplication).

Even for the simplest example $E = S^1 \times V$, this group has a non-trivial theory of unitary representations, especially, when we admit not only linear, but also projective representations. The point is that this group and its Lie algebra have remarkable central extensions. In the pure algebraic setting it was first observed by the algebraist Richard E. Block, but his result remained unobserved by mathematical physicists. Instead, they made great publicity for the so-called *Virasoro algebra*, never defined by Miguel Virasoro. The first accurate description was done by Gelfand and Dmitry B. Fuchs in the frame of the newly created homological theory of infinite-dimensional Lie algebras. This domain of modern analysis becomes popular in connection with mathematical physics, but it is also a deep and non-trivial mathematical theory. It was also the motivation for creating the general theory of hypergeometric functions. The basic idea is that the true domains of definition of these functions are the Grassmann manifolds.

Integral Geometry

The integral geometry branch of functional analysis was first introduced by Gelfand. It first appeared in the Gelfand-Naimark papers on infinite dimensional representation of classical complex Lie groups and in subsequent papers with Graev on the "orisphere method." As Gelfand himself used to say, the nice term "integral geometry" is already claimed, unfortunately, by a useful, but by no means deep, problem of calculating invariant measures on homogeneous manifolds.

Instead, he proposed as the main problem of integral geometry the study of an integral transformation T of the following type. Assume that a manifold X contains a family of submanifolds $X_y \subset X$, labeled by the points of another manifold Y. If every X_y is endowed by a measure, given by a smooth density μ_y, then to every test function f on X, the corresponding function $\varphi = Tf$ on Y is defined by

$$\varphi(y) = \int_{X_y} f(x)\mu_y(x).$$

The problem is to describe the image and kernel of T and find the inverse transformation T^{-1}, when it exists.

As an example, one can consider the classical Radon transform, where $X = \mathbb{R}^n$ and Y is the set of hyperplanes in X.

More elegant is its projective variant: X is the projective space $\mathbb{P}^n(\mathbb{R})$ and Y is the dual projective space $\mathbb{P}^n(\mathbb{R})$ of $(n - 1)$-dimensional subspaces.

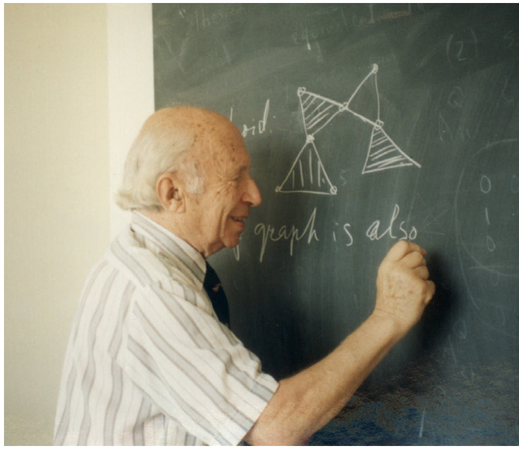
In the representation theory of complex classical groups, the pivot result, the Plancherel formula, was obtained using this remarkable fact from integral geometry: the invertibility of the orispherical transform.

In the simplest case of the group SL(2, $\mathbb{C}$), it looks as follows: a function on the 3-dimensional complex space $\mathbb{C}^3$ is uniquely determined by its integrals over all (complex) lines, intersecting with the fixed quadratic curve (e.g., the hyperbola $xy = 1$, $z = 0$).

Gelfand Mathematical Seminar

In 1943, Gelfand established his legendary Mathematics Seminar, which operated independently of the university and was open to everyone. The seminar continued for almost half a century and produced several generations of talented mathematicians.
I was a permanent member of the seminar from 1957 until its end in 1990, when Gelfand moved to the United States. There are many rumors, even legends, about this seminar. I tell here my personal impressions and add some stories and opinions of other people.

I first met Gelfand when I was a junior of twenty-one and he was a famous professor of forty-four. It was Felix A. Berezin who brought me to the seminar and introduced me to Gelfand. Probably, Berezin praised me before bringing me to the seminar. Therefore, Gelfand received me somewhat skeptically. He asked me a couple of questions about ideals in the matrix ring, f or which I did not know the answer. Turning to Berezin, Gelfand looked at him with an expression evidently saying: "whom have you brought to me?"



Gelfand teaching. (Photo by Tatiana Gelfand.)

I do not remember how Berezin got out of the difficulty, but I solved all of Gelfand's problems as soon as I returned home and on the next Monday returned to the seminar. To my disappointment, Gelfand only nodded to me remotely when I said that I had solved his problems and found them rather easy. Though Gelfand's reaction pained me, I continued to attend the seminar because a lot of new and interesting things were discussed there, always showing mathematics from different sides.

Rather quickly I became a zealous member of the seminar and acquired some permanent duties. One of them was to keep an eye on the chalk. At that time at Moscow State University, the chalk was of much better quality than in other institutes and public schools. On the other hand, the supply was sometimes insufficient, and people often took the chalk from one room to another. When the speaker was short of chalk, Gelfand used to say, looking to me, "And who is in charge on the chalk?" I then ran along to the neighboring rooms, or even to another floor, seeking the chalk.

Another duty arose a bit later and was more serious: I was nominated "a control listener" (a sort of experimental animal). The duty of a control listener was to listen to the speaker and ask questions if something was unclear. If the control listener went too long (in Gelfand's opinion) without asking questions, Gelfand might ask the listener himself to go to the blackboard and repeat what had been said until now.

Sometimes, if the control listener remained silent or said that everything is clear, Gelfand would say: "I shall nevertheless explain..." And, indeed, he occasionally offered such an explanation that all attendants (including the speaker himself) would sit open-mouthed, being surprised at how unexpectedly he could interpret the speaker's results.

It is probably worth mentioning the widespread view that Gelfand could be harsh or even brutal in his manners during the seminar. Indeed, he sometimes gave occasion for such opinion. Now, after twenty-five years of work in the United States and having visiting many other mathematical communities, I understand why most foreigners were shocked by the style of the seminar. Gelfand repeatedly broke conventional rules of academic discourse by interrupting speakers, calling participants to the blackboard, and dispatching mocking remarks.

For example, during one of my first appearances at the seminar, I was shocked by how Gelfand addressed Georgiy E. Shilov, the esteemed professor of our department. Missing, or not getting a part of a talk, Shilov addressed the speaker with some question. Gelfand interrupted and said reproachfully: "Georgiy Egenyevich! Here you listened and listened ...and understood absolutely nothing!"

To my surprise, Shilov meekly accepted the reproach and his face expressed full consent with the reproof. Only later, I learned that Shilov and Gelfand knew each other and had collaborated since 1933, when the former was a sophomore and the latter was a graduate student.

On another occasion, Gelfand asked the speaker to not distract the attendees with historical details and long explanations, but go directly to the point. When the speaker nevertheless offered explanatory remarks, he was strictly told that he must either state the result or stop the talk if he did not have a result to present.

I remember also a confrontation between Gelfand and Naimark, who was an extremely gentle and immaculately polite person. Nobody, including Gelfand, could ever imagine being rude to him. And indeed, he was not. The source of discord was the following. In talking about a recent result on the decomposition into irreducible components of the tensor product of two irreducible representations of discrete series of SL (2, R), Naimark highlighted the importance of the convergence of a certain integral. The following brief discussion ensued after Gelfand asked, "Is the integral invariant?"

"Yes," Naimark answered.

"Then, it converges," said Gelfand.

"But it needs to be proved," responded Naimark.

"No, it does not," said Gelfand, smiling and slightly waving a hand.

"Why so?" insisted Naimark.

“No need,” shrugged Gelfand, showing by his air that the further discussion did not make sense.

Later, thinking about this dialog, I understood that really, the convergence of the integral follows from the general principles, and a mathematician of such abilities as Naimark must understand it. Apparently, Naimark had obtained his result very recently and had no time to polish the proof.

With me personally, Gelfand was never rude or even hard. But some of my friends were deeply offended by Gelfand’s attitude and I am not one to judge who was right. Tanya Khovanova, the winner of several mathematical Olympiads and one of the youngest members of seminar, remembers her experience of being a control listener in the following words: “This is how it worked. The speaker starts his lecture and Gelfand interrupts him. He then turns to me and asks if I understand what the speaker just said. If I say ‘no,’ he says that I am a fool. If I say ‘yes,’ he invites me to the blackboard to explain. Usually, Gelfand finds some fault in my explanation and calls me a fool anyway. As a result, whatever I do, I end up as a fool. Ironically, I admired Gelfand for the way he conducted his seminars. I went to so many seminars where it was clear that no one understood anything. He was the only professor I knew who made sure that at least one person at his seminar– himself–understood everything.”

The regulation of the seminar was also unusual. Officially, it had to start at 7 p.m., but practically, it started with Gelfand’s appearance, which was unpredictable (usually, between 7 p.m. and 8 p.m.). Most of attendees arrived around 6:30 p.m., and it was a sort of mathematical club, where people met each other, argued, and discussed news and mathematical problems. Another special feature of the seminar was that the subject of the talk was never announced in advance. Moreover, even in rare cases, if the subject of the talk had been announced, Gelfand would change it.

The number of people at the seminar varied. Some speakers drew between 100 and 120 attendants. The average number, as I remember, was around fifty, increasing sometimes to 100, but never less than twenty or thirty. The left side of several front rows was occupied by young students: freshmen, sophomores, even high school students, working with Gelfand. I usually was on the right side among other regular attendants. Starting in the fifth or sixth row were different people, grouped by interest, age, and speciality.

Gelfand about Himself and about Mathematics

For the jubilee anniversary of Moscow High School Number 2 in 2006, Gelfand addressed the students with the following words:

> *I myself learned a lot during my work in this school. I understood that one cannot be interested in mathematics only and that mathematics is not a sport….Mathematician is someone who understands. It is necessary not only to have the skill to solve difficult problems, but to understand mathematics….I want to name four important features, common for mathematics, music and other sciences and arts: the first–the beauty, the second–the simplicity, the third–the exactness and the fourth–the crazy ideas.*

In his response to winning the Steele Prize for Lifetime Achievement from the AMS, Gelfand wrote:

> *Mathematics for me is a universal and adequate language of sciences, and it is an example of how people of different cultures and backgrounds can communicate and work together. This is extremely important in our times.*

On the eve of his ninetieth birthday, Gelfand reflected on the essence of mathematical achievement: "It is not only about aptitude," he said, sitting in his cozy office on the Busch campus of Rutgers University in Piscataway. "It is about appetite. In our century of rapid changes, it is impossible to know everything. The goal is to learn how to learn."

Others on Gelfand

It is well known that Gelfand was on bad terms with many mathematicians, including very influential ones. The opinions of students and colleagues regarding this aspect of Gelfand's professional life are rather contradictory. I already said that with me and with most of my friends he was never rude or offensive. But some others think differently. The following are some comments from known mathematicians that were published or pronounced in public.

Michael Tsetlin (during a banquet in honor of I. M.): "Israel Moiseyevich, I know why you are an honorable member of many Academies and Societies abroad, while in our country you are even not a full member of Academy! Why? Because there you are known only by your papers, while here also personally."

Terence Tao (Fields Medal): "I met Gelfand only once, in one of the famous Gelfand seminars at the IHES in 2000. The speaker was Tim Gowers, on his new proof of

Szemeredi's theorem (Endre Szemeredi, incidentally, was Gelfand's student). Gelfand's introduction to the seminar, on the subject of Banach spaces which both mathematicians contributed so greatly to, was approximately as long as Gowers's talk itself!"

Academician Andrey N. Vorob'ev (Director of the National Research Center for Hematology of the Russian Academy of Sciences and former student of Gelfand): "What can be said? To say that he is a boor is to say nothing. He is disgracefully heel."

Vladimir M. Tikhomirov (Professor at Moscow State University): "I stopped attending Gelfand's seminar for two reasons. First, I did not understand many things. Second, Gelfand's attitude to the audience was by no means delicate. Once, near the room 14-08 (the seminar place) I saw my friend, a very talented young mathematician, who was very excited. When I asked him, what is the matter, he rushed on me a squall of damnation, which he prepared for Gelfand when he goes out. The point was that my friend happened to ask a question to the speaker. Gelfand exclaimed: 'Do not answer him. Our seminar is for literate people.' I hardly calmed my friend, but decided not to go to the seminar to avoid similar remarks in my address."

Volodya Gelfand: "I. M. knew no biology, but was always able to identify true experts to talk, and these discussions were often very beneficial for the biologists as well."

Dima Leshchiner: "I recall his favorite saying: 'People do not have shortcomings, but only peculiarities.' It seems to me this has to do with what 'decency' meant in his understanding, namely, that 'decency' is the quality of an action, not of a person."

A. A. Beilinson: "I. M. loved playing with people (with him mischief was never far away). A common way to engage someone was to explore his feeling of self-importance. I. M. rarely lost the game; if this happened (which meant that the opponent was more unpredictable than I. M. himself), he was furious, but the winner got his respect and, perchance, even love. For example, I. M. could ask you to wait and then disappear for a very long time. A cheap win was to leave after an hour. A master stroke would be different. According to legend, when I. M. returned to his office after several hours to see how Misha Tsetlin was doing, he found Misha fast asleep on I. M.'s sofa."

Spencer Bloch: "I am sure I told you my Gelfand story when he came to Paris and was to meet with Serre. He was staying at Ormaille and the people at IHES needed someone to escort him to Paris. I was elected. I suggested we take a train with plenty of time to spare so we would not inconvenience the great Serre. Of course, I did not fully grasp the subtle thinking process of my charge. Suffice it to say that not inconveniencing Serre was rather low on the totem pole of Gelfand's priorities. I arrived at his apartment and he announced that he would instruct me on the Russian technique for making tea. So, of course, we missed the train. But I said no matter, there would be another train along in 20 minutes. But no, Gelfand said that errors had occurred during the making of the tea, and nothing would do except to return to his apartment and make more tea; which we did. So, of course, we missed the next train. And, as was clearly the intent from the beginning, the great Serre was made to wait for the great Gelfand."

V. I. Arnold: "I heard from my teacher A. N. Kolmogorov, that in the presence of only two persons he felt the intellect equal or bigger than his own ('sensed the presence of higher mind'). One of them was I. M. Gelfand."

A. V. Alexeevski: "Perhaps, the right word about I. M. is 'the search of the harmony'."

S. G. Gindikin: "I. M. could be put in the Guinness Record Book as the person who actively worked in mathematics longer than anybody: 74 years, from 20 to 94."

A. B. Sossinski, the vice dean of the IMU (Reminiscence about Gelfand's seminar on the site Polit.ru): "Gelfand organized, probably, the best mathematical seminar in the history of mathematics."

N. Ya. Mandelstam, the widow of the poet, said once: "You need not to be mathematician to understand that Gelfand is a genius."(The site Polit.ru)

E. G. Glagoleva (One of organizers of the Gelfand Correspondence School): "Everyone who communicated with I. M. has become his student."

Awards and Honors

In his lifetime, I. M. Gelfand received many awards, among them the two Stalin Prizes (1951, 1953), three Lenin orders (1954, 1956, 1973), the Lenin Prize (1961), the Wolf Prize (1978), the Wigner Medaille (1980), the Kyoto Prize (1989), a MacArthur Fellowship (1994), and the State Prize of the Russian Federation (1997). In 1989, he was invited to give the Silliman Lecture (1989), and the Bowen Lecture in 1992-93. He was elected an International Member of the National Academy of Sciences in 1970 and the Royal Society in 1979. In 2005, he received the Steele Prize for Lifetime Achievement from the AMS.

Gelfand served as the president of the Moscow Mathematical Society from 1968 to 1970. He was elected an honorary member of the American Academy of Arts and

Sciences, the Royal Irish Academy, the American Mathematical Society, and the London Mathematical Society. He has been awarded many honorary doctorates, including one from the University of Oxford. He was elected as a plenary speaker at International Congress three times (1954, 1962, and 1970) but never received permission to attend. He also established the Correspondence Program in Mathematics (GCPM) at Rutgers in 1990.

From 1958 onwards, Gelfand became interested in problems in biology and medicine. In 1960, together with Fomin and other scientists, he set up the Institute of Biological Physics of the USSR Academy of Sciences. In particular, he became interested in cell biology and experimental work as well as the theoretical work that was his first interest. In an article by J. J. O'Connor and E. F. Robertson in September 2009, his work in biology was described as follows:

> *On the basis of actual biological results, he developed important general principles of the organization of control in complex multi-cell systems. These ideas, apart from their biological significance, served as a starting point for the creation of new methods of finding an extremum, which were successfully applied to problems of X-ray structural analysis, problems of recognition, etc.*

Israel Moiseivich Gelfand has 26 PhD students and 682 descendants.

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