Herbert Federer

1920–2010

BIOGRAPHICAL *Memoirs*

A Biographical Memoir by
Wendell H. Fleming
and William P. Ziemer

NATIONAL ACADEMY OF SCIENCES

HERBERT **FEDERER**

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Elected to the NAS, 1975

By Wendell H. Fleming and William P. Ziemer

Herbert Federer was born in Vienna, Austria, in 1920, immigrated to the United States in 1938, and became a naturalized citizen in 1944. Federer was a rather private person. In later years, he was reluctant to discuss with colleagues the painful memories of events related to his departure from Austria. For the rest of his life, he chose never to travel to Europe.

Federer began his college education at a teachers college which later became the University of California, Santa Barbara. His exceptional mathematical talent was quickly recognized, and he soon transferred to the University of California, Berkeley. He received from Berkeley a B.A. in mathematics and physics in 1942 and his Ph.D in mathematics in 1944. During 1944 and 1945 Federer served in the U.S. Army at the Ballistics Research Laboratory in Aberdeen, MD. Immediately after receiving an Army discharge, he joined the Department of Mathematics at Brown University, where he remained until his retirement in 1985. Another major event in Federer's life was his marriage in 1949 to Leila Raines, who survives him at this writing. Mathematics and his family were Herb's two great loves. He was devoted to his wife and their three children.

Federer joined the American Mathematical Society (AMS) in 1943. He served as associate secretary during 1967 and 1968 and as AMS representative to the National Research Council from 1966 to 1969, and at the 1977 summer AMS meeting in Seattle he was the Colloquium Lecturer. In 1987, he and Wendell Fleming received the AMS's Steele Prize for their 1960 paper "Normal and integral currents."

Federer was an Alfred Sloan Research Fellow (1957–1960), a National Science Foundation Senior Postdoctoral Fellow (1964–1965), and a John Guggenheim Memorial Fellow (1975–1976). He became a fellow of the American Academy of Arts and Sciences in 1962 and a member of the National Academy of Sciences in 1975. He passed away on

April 21, 2010. An article about his life and career appeared in the May 2012 issue of the *Notices of the AMS*.[1]

Federer is remembered for his many deep and original contributions to the fields of surface area theory and geometric measure theory. It is difficult to imagine that the rapid growth of geometric measure theory (beginning in the 1950s), as well as its subsequent influence on other areas of mathematics and applications, could have happened without his groundbreaking efforts. Research on surface area theory flourished from the 1930s through the 1950s, and beginning with two 1944 papers Federer quickly became a leader in that field (Federer 1944a, 1944b) [See Selected Bibliography below]. He developed techniques, based on new methods in algebraic topology, that were applicable to area theory for surfaces of any dimension k —and not merely for k=2, as in previous work.

One of the goals of geometric measure theory is to provide a theory of k- dimensional measure for subsets of n-dimensional Euclidean space $\mathbb{R}^n$, for any dimension $k \leq n$. Federer's definitive 1947 paper described the structure of subsets of $\mathbb{R}^n$ that have finite k-measure in the sense of Hausdorff. Another goal is a theory of integration over k-dimensional "surfaces," which are defined in a suitable weak sense. Federer's 1945 paper was an initial step, in which the classical Gauss-Green theorem was obtained without traditional smoothness assumptions. In later work, he defined surfaces of dimension k as integral currents; his 1960 paper with Fleming stimulated the development of geometric measure theory during the 1960s and afterward. Federer's authoritative book *Geometric Measure Theory* appeared in 1969. To quote his former Ph.D. student Robert Hardt:

> *Forty years after the book's publication, the richness of its ideas continue to make it both a profound and indispensable work. Federer once told me that, despite more than a decade of his work, the book was destined to become obsolete in the next 20 years. He was wrong. The book was just like his car, a Plymouth Fury wagon purchased in the early 1970s that he somehow managed to keep going for almost the rest of his life. Today [May 2012], the book Geometric Measure Theory is still running fine and continues to provide thrilling rides for the youngest generation of geometric measure theorists.*[2]

1 Parks, H. 2012. Remembering Herbert Federer (1920-2010). *Notices Amer. Math. Soc.* 59:622–631.
2 Ibid., 626.

Federer's 1978 paper on the subject was based on his 1977 AMS Colloquium Lectures. Nonspecialists may find it a useful complement to the more detailed development in his 1969 book.

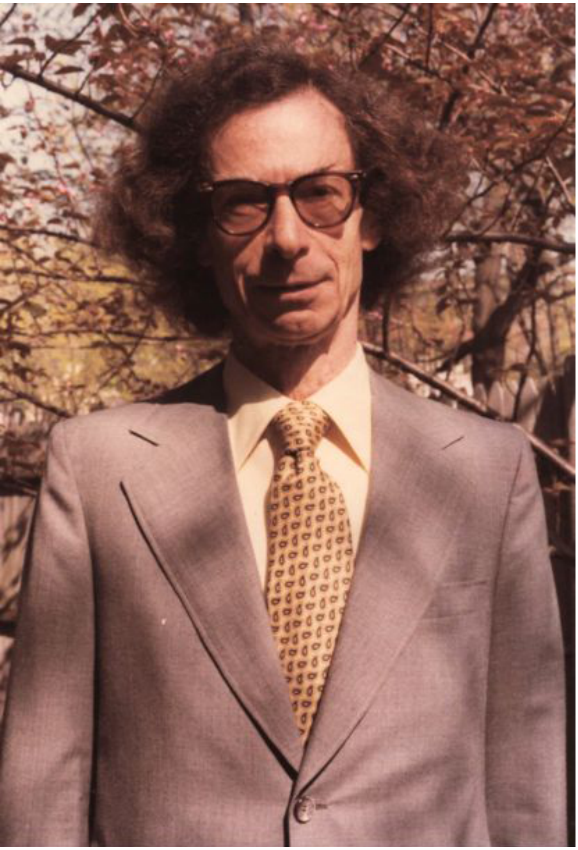
(Photo courtesy Department of Mathematics. Brown University.)

Both surface area theory and geometric measure theory were partly motivated by geometric problems in the calculus of variations. Plateau's Problem (or problem of least area) is a typical example. Federer's results of 1960 imply the existence of k-area minimizing integral currents that have given (k-1)-dimensional boundary (Federer 1960). There remained the very difficult problem of regularity of such k-area minimizing currents, for which Fred Almgren (who was Federer's Ph.D. student), Federer himself (Federer 1970), and others contributed results.

Federer's 1965 paper provided insights that created new linkages between Riemannian, complex, and algebraic geometry through what is now called calibration theory. He showed the following: Let M be a Kähler manifold of complex dimension m, and V any complex variety in M of complex dimension k. Then V corresponds to a locally $2k$-area minimizing integral current. By taking M to be complex Euclidean space $C^m = \mathbb{R}^{2m}$, a large class of examples of singular sets for $2k$-area minimizing integral currents in $\mathbb{R}^{2m}$ is obtained.

A further discussion of Federer's research contributions, including some technical details, is given later in this article.

One characteristic of Federer's work was his dedication to learning many different kinds of mathematics. When he became interested in a new subject (e.g., algebraic topology, differential geometry, or algebraic geometry), he would first spend many weeks reading classical and modern books on it. He would then teach a graduate course on the topic and produce a large collection of lecture notes. One of his principal points of advice to graduate students was indicated by the only sign on his office door. It was a long, vertically stacked series of small stickers that said "Read, Read, Read, ..."



(Photo courtesy Department of Mathematics. Brown University.)

Federer set very high standards for his mathematical work and expected high-quality research from his students; he supervised the Ph.D. theses of 10 of them. As of January 2013, the Mathematics Genealogy Project website listed for him 169 Ph.D. mathematical descendents (students, "grandstudents," and "great-grandstudents"). While some students found Federer's courses daunting, he was very welcoming to anyone who was deeply committed to mathematics and who took the trouble to get to know him.

Fair-minded and very careful to give proper credit to the work of other people, Federer also was generous with his time when serious mathematical issues were at stake. He was the referee for John Nash's 1956 *Annals of Mathematics* paper "The imbedding problem for Riemannian manifolds," which involved a collaborative effort between author and referee over a period of several months. In the final accepted version, Nash stated, "I am profoundly indebted to H. Federer, to whom may be traced most of the improvements over the first chaotic formulation of this work." This paper provided the solution to one of the most daunting and longstanding mathematical challenges of its time.

Federer's research contributions

Federer's work can be grouped into two categories: Lebesgue area theory and geometric measure theory (GMT), although there is considerable overlap between the two. Federer was considered a giant by many mathematicians because of his profound influence on geometric analysis. The first published paper of his career was the result of his asking

A. P. Morse, who later turned out to be his Ph.D. mentor, for a problem to test whether he was capable of being a research mathematician. In Federer's resulting joint paper on GMT with Morse (1943), the answer became abundantly clear that Federer indeed had the right stuff.

Federer, one of the creators of GMT, is perhaps best known for his fundamental development of the subject throughout his career, which culminated in the publication of a treatise (Federer 1969). This book, nearly 700 pages long, has become the standard-bearer for GMT, and it is written in a manner that commands both admiration and respect because of its virtually flawless presentation of a wide range of mathematical subjects. Morever, it is written in a style that is unique to Federer. The book was carefully prepared in handwritten notes and includes an extensive bibliography of approximately 230 items. Such attributes are typical of all of his writings.

The problem of what should constitute the area of a surface confounded researchers for many years. In 1914 Carathéodory defined a k-dimensional measure in $\mathbb{R}^n$ in which he proved that the length of a rectifiable curve coincides with its one-dimensional measure. In 1919, Hausdorff, developing Carathéodory's ideas, constructed a continuous scale of measures. After this, it became obvious that area should be regarded as a two-dimensional measure and should establish the well-known integral formulas associated with area. Later, Lebesgue's definition (somewhat modified by Frechet) of area—as being the lower limit of areas of approximating polyhedra—became dominant, partly because of its successful application in the solution of the classic Plateau's (least-area) Problem. This definition had the notable feature of lower semicontinuity, which is crucial in the calculus of variations. Federer's two papers of 1944 mark the beginning of his research on Lebesgue area, a field that until that time was led by two influential mathematicians, Lamberto Cesari and Tibor Rado. In Federer (1944b), he considered the problem that is perhaps the central question in area theory, the answer to which had been sought by many researchers.

Basic question in area theory: Let f denote a continuous map from U into $\mathbb{R}^n$, where U is a region in $\mathbb{R}^k$. What is the type of multiplicity function that, when integrated over the range of f with respect to the Hausdorff measure, will yield the Lebesgue area of f?

In Federer (1944b), his results imply that if all the partial derivatives of f exist everywhere in a region T, then the Lebesgue area can be represented as the integral of the crude multiplicity function $N(f, T, y)$ which denotes the number of times in T that f takes the value y.

Federer's paper of 1947 really lays the foundation for the development of GMT. Up to the time of this paper, A. S. Besicovitch had studied the geometric properties of plane sets of finite Carathéodory linear measure, and these studies were extended by A. P. Morse and J. F. Randolph. The corresponding problems for two-dimensional measures over three-dimensional space are connected with the theory of surface area. This paper contained a discussion of these properties for a large class of k-dimensional (outer) measures over n-dimensional space and also developed some of the fundamental tools of GMT. For example, Federer showed that any set $E \subset \mathbb{R}^n$ with finite k-dimensional Hausdorff measure can be decomposed into rectifiable and non-rectifiable parts. Then Federer applied the preceding theory to show that the Hausdorff measure of a two-dimensional nonparametric surface in $\mathbb{R}^3$ equals the Lebesgue area of the map defining the surface. In a measure-theoretic sense, the rectifiable part of E almost coincides locally with the graph of a Lipschitz function. For most projections of $\mathbb{R}^n$ onto a k-plane through the origin, the unrectifiable part of E projects onto a set of k-dimensional Lebesgue measure 0.

The problem of finding a suitable multiplicity function such that its integral over the range of f will yield the Lebesgue area of f remained intractable until Federer brought some notions of algebraic topology to bear. In Federer (1946), an area is defined for all continuous k-dimensional surfaces in terms of the stable values of their projections into k-dimensional subspaces; the area thus defined is lower semicontinuous. Its relation to Lebesgue area is only partially settled in this paper. Then, in Federer (1948), results were announced that represented generalizations to n dimensions of material previously known only in the two-dimensional case. The topological index, which had been used as a principal tool in the two-dimensional case, was replaced by the topological degree (expressed in terms of Čech cohomology groups) and by the use of the Hopf Extension Theorem. These changes allowed the stable multiplicity function to be determined by merely counting the number of essential domains of $f^{-1}(U)$, where U is a domain in $\mathbb{R}^n$. The techniques of algebraic topology were fully applied. The key to extending the theory of Lebesgue area from two-dimensional surfaces in $\mathbb{R}^3$ to surfaces in $\mathbb{R}^n$ was the generalization of Cesari's inequality from $\mathbb{R}^3$ to $\mathbb{R}^n$.[3] That inequality states that the Lebesgue area of a mapping $f{:}X \to \mathbb{R}^3$ is dominated by the sum of the areas of its projection onto the three coordinate planes. Here X denotes a finitely triangulable subset of the plane. In his paper of 1955, Federer proved the extension of this inequality to $\mathbb{R}^n$, which was a monumental achievement as it necessitated the complete development of the length

3 Cesari, L. 1942. Caratterizzazione analitica delle superficie continue di area finita secondo Lebesgue. *Ann. Scuola Norm. Super. Pisa* (2) 11:1–42.

of light mappings defined on an arbitrary metric space, thus foretelling the directions of modern day GMT. Here, the length of a light mapping $f{:}X \to Y$, where X is assumed to be a locally compact, separable metric space and Y an arbitrary metric space, is defined as the supremum of $\sum \mathrm{diam}[f(C)]$, where the supremum is taken over all countable disjoint families of nondegenerate continua in X. So, with this result, the theory of Lebesgue area for surfaces in $\mathbb{R}^3$ can be essentially generalized to surfaces in $\mathbb{R}^n$.

Federer's paper of 1955 was one of his best efforts in area theory. In particular, it contained the basic idea that led to the fundamental result, the Deformation Theorem of GMT (Federer and Fleming 1960). The paper appeared in the *Annals of Mathematics* because it was rejected for publication by the *Transactions of the AMS* despite the fact that it was of the highest quality. This bothered Federer considerably, and he contemplated leaving the field. Fortunately, he did not, for his best work was yet to come. For example in his and Demers' paper of 1959 (1959b), the authors went on to improve the results of Federer (1955) by showing that in the case of a flat mapping f, a mapping in which both the domain space and the range space are of the same dimension, the k-dimensional Lebesgue area of f equals the integral of a new multiplicity function that is defied in terms of norms of cohomology classes.

The paper Federer (1959a) establishes a very useful result in GMT, known as the co-area formula. In its most elementary form, it states that if f is a real-valued function of class C^1, then the total variation of f can be expressed in terms of integration of f over the fibers of f with respect to (n-1)-dimensional Hausdorff measure. In its more general form, the formula is valid for any Lipschitz mapping from X to Y, where X and Y are separable Riemannian manifolds of class 1 with respective dimensions n and k, $n \geq k$. This result has generated great interest and has led to many applications and generalizations. For example, Fleming and Rishel (1960)[4] established a co-area formula for $f \in BV(\mathbb{R}^n)$, while Malý, Swanson, and Ziemer (2003) proved it for a suitable class of Sobolev mappings.[5]

As for the basic question in area theory that was stated above, the answer was provided in Federer's last publication on the subject (1961). Let $f{:}X \to \mathbb{R}^n$ be a continuous mapping, where X is a compact manifold of dimension $k \leq n$ Assuming that f has finite Lebesgue

4 Fleming, W. H., and R. Rishel. 1960. An integral formula for total gradient variation. *Arch. Math. (Basel)* 11:218–222.

5 Malý, J., D. Swanson, and W. P. Ziemer. 2003. The co-area formula for Sobolev mappings. *Trans. Amer. Math. Soc.* 355(2):477–492.

area and that either k=2 or that the range of f has $(k+1)$-dimensional Hausdorff measure 0, Federer proved that there exists a unique current-valued measure μ defined over Mf, the middle space associated with f, such that the total variation of μ is equal to the Lebesgue area of f. Moreover, the density of μ, with respect to k-dimensional Hausdorff measure, yields a multiplicity function that provides the answer to the basic question. While Federer was writing this 1961 paper, he said that he intended to write it very concisely because he knew that area theory was a dying field and that the paper would not generate much interest. By that time, he was already consumed with the development of GMT.

Even for the casual reader of Federer's work, it becomes clear that he brought an incredible arsenal of tools to bear on whatever the problem at hand. It is also clear that his determination to learn virtually everything about a problem was highly unusual—for example, his taking a period of 17 years to answer the fundamental question cited above. In reviewing Federer's 1946 paper, G. Bailey Price apparently agreed:

> *The paper as a whole is characterized by the treatment of problems and the employment of methods of great generality. The author uses many results from two of his previous papers [Federer 1944a, 1944b]. In addition, he employs a wide variety of powerful tools selected freely from the theory of topological groups, measure theory, integration theory, the theory of functions of real variables, topology, and other fields of modern mathematics.*

As an indication of how he has inspired others to carry on his work, consider that the number of citations in *Mathematical Reviews* to his book on GMT is nearly 1,500, and consider as well the recent work of those who have extended Federer's work to metric spaces.[6, 7]

As indicated above, Federer made contributions to GMT in his writings (Federer 1947, 1955, 1959a and b, 1961), but it was the seminal paper "Normal and integral currents" (Federer and Fleming 1960) that marked the birth of GMT and went on to receive the American Mathematical Society's 1987 Steele Prize. The paper was devoted to the development of the notion of a generalized k-dimensional surface in $\mathbb{R}^n$, which

6 Ambrosio, L. 2001. Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces. *Adv. Math.* 159(1):51–67.

7 Ambrosio, L. 2002. Fine properties of sets of finite perimeter in doubling metric measure spaces. *Set-Valued Anal.* 10(2–3):111–128.

would include a powerful compactness property—achieved by employing de Rham's currents, which when $k = 0$ become Schwartz distributions. Because a current is defined as a continuous linear functional on a space of differential k forms, there is a natural definition of boundary, namely $\partial T := T \circ d$, where d denotes the exterior derivative of a k-form φ. This formulation facilitates the study of homology groups of spaces of integral currents. The mass $M(T)$ of a k-dimensional current T is its norm when considered as a linear function of continuous k forms and a current T for which $N(T) := M(T) + M(\partial T) < \infty$ is called normal. The mass closure of the space of Lipschitz images of Lipschitz chains with integer coefficients are called "rectifiable currents." If both T and ∂T are rectifiable, T is called an integral current." It turns out that a current T is rectifiable if and only if there exist a k-rectifiable set $E \subset \mathbb{R}^n$ and Hausdorff measurable functions Θ and M defined on E such that M is integer-valued and that $T(\varphi) = \int_E M(\varphi \cdot \Theta)\, dH^k$ for all k-forms φ. That the space of integral currents was destined to be an object of great interest in the calculus of variations was due to the powerful closure theorems that showed the weak limit of a sequence of N-bounded integral currents to be an integral current and every normal rectifiable current to be an integral current. These results have immediate applications to the Plateau's Problem, for they imply the existence of k-dimensional rectifiable area-minimizing varieties with prescribed boundaries in $\mathbb{R}^n$.

The next question that needs to be addressed is the smoothness of the area-minimizing current T. This is called the regularity question, which turned out to be notoriously difficult. The goal was to prove smoothness of the support of a k-area-minimizing integral current T, except at points of a singular set of lower Hausdorff dimension. Examples show that in dimensions $1 < k < n-1$, the singular set can have Hausdorff dimension k-2. Eventually, Almgren proved that the singular set in fact has dimension of at most $k-2$. This was a true tour de force in that it represented a 10-year effort and resulted in a 1,700-page manuscript.

In codimension 1 ($k = n-1$), it seemed at first that area-minimizing currents might have no singular points. This turned out to be correct for $n \leq 7$ by results of DeGiorgi, Almgren, and Simons. However, in 1969 Bombieri, DeGiorgi, and Giusti gave an example of a seven-dimensional cone in $\mathbb{R}^8$ that provides a seven-dimensional area-minimizing integral current with a singularity at the vertex. In Federer's paper of 1970 he showed that this example is generic in the sense that the singular set can have Hausdorff dimension at most n-8. In the same paper Federer also considered another version of Plateau's Problem, which in effect ignores orientations. He showed that, for this "non-oriented version," the singular set has Hausdorff dimension of at most k-2 for arbitrary k. Thus, in the nonoriented case, Federer was able to obtain the optimal regularity result much more directly.

Normal currents were shown to have many applications, most notably in the development of functions defined on an open set $U \subset \mathbb{R}^n$ whose derivatives are measures $BV(U)$. Federer proved that a function $f \in BV(\mathbb{R}^n)$ with compact support corresponds to a normal current of dimension $k=n$. Because of Theorem 4.5.9 in his 1969 book—a theorem with 31 parts to it—the class of functions on whose partial derivatives are representable by integration is now widely regarded as the proper generalization to $n>1$ of the class of those functions on $\mathbb{R}$ that are equal almost everywhere to functions of locally bounded variation. Then, despite the fact that a BV function can be discontinuous everywhere, results are given that use H^{n-1}-approximate upper and lower limits to provide a complete extension to $n>1$ of the classical results describing the continuity properties of functions of bounded variation. The extension includes that f can be defined H^{n-1}-almost everywhere as the limit of its integral averages). Furthermore, it is shown that the set

$$G := (\mathbb{R}^n \times \mathbb{R}) \cap \{(x, y) : \lambda(x) \leq y \leq \mu(x) \tag{1}$$

where $\lambda(x)$ and $\mu(x)$ denote the lower and upper approximate limit of f at x, is n-rectifiable.

Research on the problem of finding the most natural and general form of the Gauss-Green theorem has contributed greatly to the development of geometric measure theory. Throughout the proof of Theorem 4.5.9, a prominent role was played by the Gauss-Green theorem of De Giorgi and Federer.[8] The classical Gauss-Green theorem states that if $E \subset \mathbb{R}^n$ is a bounded set with smooth boundary B and V is a smooth vector field on $\mathbb{R}^n$, then

$$\int_E div V(x) dx = \int_B V(y) \cdot \nu(y) dH^{n-1}(y) \tag{2}$$

where $\nu(y)$ is the exterior unit normal at y and H^{n-1} is Hausdorff measure in dimension $n<1$. In De Giorgi and Federer's results, much weaker assumptions are made about E. If E is identified with the corresponding current T_E of dimension n, it suffices to assume that T_E is an integral current. Then equation (2) holds, with B replaced by B^*, where $B^* \subset B$ is the "reduced boundary" in De Giorgi's terminology and $\nu(y)$ is an approximate normal at y defined in a suitable measure-theoretic sense.

8 De Giorgi, E. 1955. Nuovi teoremi relativi alle misure (r–1)-dimensioni in uno spazio ad r dimensioni. *Ricerche Mat.* 4:95–113.

In another influential paper (1965), Federer defined slicing of a normal current T in a differentiable manifold X by a locally Lipschitzian map $f\colon X \to \mathbb{R}^n$. With almost every point $y \in \mathbb{R}^n$ there is associated, by means of relative differentiation of measures, a normal current $<T, f, y>$ of dimension $dim(T)-n$, which can be thought of as the slice of T in $f^{-1}y$. In case T is integral, $<T, f, y>$ will also be integral. Another important result of this paper is a new measure-theoretic characterization of integral currents. Suppose T is a k-dimensional current in $\mathbb{R}^n$ and U , V are k–-dimensional currents with $\partial T = U + V$. If T is rectifiable, $M(U) < \infty$, and $H^{k-1}(sptV)=0$, then $\mathrm{M}(\partial T) < \infty$; hence T is an integral current.

Herbert and Lila Federer.
(Photo courtesy Department of Mathematics. Brown University.)

Then Federer gives a very short proof of Wirtinger's inequality, from which it follows that on a Kähler manifold, integral currents with almost-everywhere complex tangent spaces are minimal currents. Using the characterization of integral currents above, one sees that each complex algebraic variety of complex dimension k on a Kähler manifold X is naturally a minimal (locally) integral current of dimension $2k$ on X. The following striking fact is a consequence. In the higher-dimensional versions of the Plateau's Problem (now formulated in terms of minimal currents), not only will singularities ordinarily arise in the solutions, but in particular every singularity of a complex algebraic variety can occur.

Federer saw that his theory of slicing would also have application to intersection theory, a concept that was initiated by Kronecker.[9] In fact, in his 1969 book Federer conjectured that his theory of slicing could be used to construct a viable intersection theory for real

9 Kronecker, L. 1868. Über systeme von functionen mehrerer variabeln. In *Monatsber. König. Preuss. Akad. Wiss. Berlin*. 339–346.

analytic chains. This conjecture was validated by Robert Hardt,[10] a Ph.D. student of Federer, when he showed the following:

Let n, $t \geq n$ be integers, M a seperable oriented real anlytic manifold, T a t-dimensional analytic chain in M, f an analytic map from M into the n-dimensional Euclidian space $\mathbb{R}^n$,$<T, f, y>$ the slice of T in $f^{-1}\{y\}$ for almost all $y \in \mathbb{R}^n$, and $Y = \{y \in \mathbb{R}^n : dim(f^{-1}\{y\} \cap sptT) \leq t - n$ and $dim(f^{-1}\{y\} \cap spt\partial T) \leq t - n - 1\}$. Then the association of $<T, f, y>$ with y maps Y into the $(t - n)$-dimensional analytic chains in M and is continuous with respect to the topology of the locally integral flat chains in M. Using this as a basic tool, Hardt then showed that if S and T are analytic chains in M and the dimensions of $sptS$ and $sptT$ satisfy certain conditions, then the intersection of S and T, denoted by $S \cap T$, is well defined by slicing the Cartesian product $S \times T$, in any coordinate neighborhood, by the subtraction map. The resulting real analytic intersection theory is then characterized by certain algebraic formulae.

Herbert Federer will be remembered as a pioneer in geometric analysis—as one who made fundamental and profound advances in many different ways, thus helping other new ideas to develop. Federer told one of us (Ziemer) while he was writing his book that he "was inscribing his epitaph on his tombstone." Indeed he did; and it is our good fortune that he did it so indelibly. His legacy will be a source of inspiration far into the future.

10 Hardt, R. M. 1972. Slicing and intersection theory for chains associated with real analytic varieties. *Acta Math.* 129:75–136.

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