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GILBERT AMES BLISS

1876—1951

A Biographical Memoir by

E. J. MCSHANE

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TODAY THERE IS a thickly inhabited part of the South Side of Chicago where the word "Kenwood" is still to be seen as the name of an avenue and of a few shops. Eighty years ago this region was the quiet, conservative suburb Kenwood of the rapidly growing city Chicago. In this suburb Gilbert Ames Bliss was born, on May 9, 1876. His father had long been associated with various electrical enterprises, and shortly after 1880 became president of one of the early Chicago Edison companies. Thus through his childhood the young Gilbert Ames heard much about the new and revolutionary inventions of the day, and soon acquired what was to be a lifelong interest in scientific subjects.

Another abiding interest had its roots in those youthful days in Kenwood. While Gilbert Ames was still a schoolboy his older brother had become prominent in bicycle racing, which then was far more popular and fashionable in the United States than it now is; at one time he held two world's records in this sport. Naturally the whole family took an interest in competitive sports, and Gilbert Ames retained this interest all his life. In college he was a member of the track team, competing in the now vanished event of bicycle racing. Later he played tennis, and still later he was an enthusiastic golfer until age and illness forbade it.

In grammar school and high school he was rather precocious, without being a prodigy. When he entered the University of Chicago in 1893, with the second entering class, he had no strong choice of direction in his studies, but with the study of calculus his interest in mathematics was aroused, and in a rather queer way. The instructor was given to severity and sarcasm, and daily brought his students to wrath or tears according to sex and disposition. A few of the students found that by hard study they could withstand him, and thus Bliss came to learn calculus well and to find it fascinating. Presumably the instructor's contribution to mathematics was positive, since it is unlikely that a potential equal of Bliss was among the others who finished with an incurable dislike for mathematics. Moreover, his recommendation helped Bliss to obtain a scholarship for the Senior College, which was important because his father's increasing age and uncertain health, together with the depression of the nineties, had made the financial situation of the family somewhat cramped.

The need of earning a good part of his expenses in Senior College was a slight handicap scholastically, but he adjusted to it and managed to do very well with his studies and to join the track team and the glee club. This last activity was related to his way of earning money; he was a member of a professional mandolin quartet. It must have been a good one, for it was financially profitable and Bliss himself felt that they played well.

Bliss's first graduate work (1897) was centered around mathematical astronomy, under the inspiring guidance of F. R. Moulton. In this year he wrote his first published paper on "The motion of a heavenly body in a resisting medium." Nevertheless, his application for a fellowship was unsuccessful. This is the sort of mistake on the part of a fellowship committee that seems especially ludicrous to those who have never served on a fellowship committee. It did not stop Bliss, but it did make him reconsider his plans. The result was a recognition that it was mathematics that held most attraction for him, and a decision to study for a doctorate in mathematics. His later research indicates clearly that he always retained a liking for the kind of mathematics that at least conceivably has applications.

The leaders in mathematics at the University of Chicago in those days were the famous three, E. H. Moore, O. Bolza, and H. Maschke.

All three of them served as inspiring teachers and as fine examples of scholars. However, it was Bolza who most influenced Bliss, both directly by his lectures and indirectly by letting him make a copy of Bolza's own record of the famous course of lectures on the calculus of variations given by Weierstrass in 1879, and then available only in manuscript form. It was under Bolza's guidance that Bliss wrote his doctoral dissertation, on the geodesics on an anchor ring.

After receiving his doctorate (1900), Bliss took up his first regular teaching position, at the University of Minnesota. He enjoyed the teaching and the students from the start; but his principal scientific activity was a self-disciplined study of a large part of the available literature on the calculus of variations. He felt that he profited so much by this program of mastery of the subject that he consistently recommended it to students, lightly, as when one of my fellow-students admitted that he had read only one presentation of a subject, and Bliss advised him to "read two books on it, then you'll be an expert"; or seriously, in private conversation; but always earnestly.

In 1902 Bliss was able to go to Göttingen for a year of further study. This was in the time of Hilbert, Klein, and Minkowski. But Göttingen offered other advantages besides the lectures and seminars of these leaders. There was the opportunity to become acquainted with the five young mathematicians Abraham, Carathéodory, Fejer, E. Schmidt, and Zermelo. There was the experience of speaking several times in seminars and in the Gessellschaft; on this last occasion he horrified the audience by a slip in language, answering Klein's blunt question "Who would be interested in this?" with "Wenn Sie etwas von Variationsrechnung kennten . . ." instead of "Wenn man etwas . . . ." There was the library, with records of courses by Weierstrass, Hilbert, Sommerfeld, and Zermelo. And, too, there was the opportunity of comparing the results of American and German teaching systems.

For the next year he chose to accept E. H. Moore's offer of a one-year assistantship at the University of Chicago, for whose department of mathematics he had a respect enhanced by his year abroad. This was followed by an enjoyable year at the University of Missouri, from which he was attracted by an invitation to join the Princeton staff. He was one of the "preceptors" (roughly the equivalent of assistant professors) added in 1905 to carry out Woodrow Wilson's plan for the reorganization of education at Princeton, and so he was a spectator of the struggles between Wilson and his opponents that finally forced Wilson to leave the faculty and begin a new and (I am told he called it) a "less political" career.

In the summer of 1908 Maschke died, and Bliss was invited to the University of Chicago as associate professor to replace him. From 1908 until his retirement he remained on the faculty of the University of Chicago. For two years he gave the advanced courses in geometry which Maschke would have given. But in 1910 Bolza resigned from the department, Wilczynski was added to the staff, and Bliss was able to return to his favorite field, analysis in general and the calculus of variations in particular.

Probably it was his earlier career in sports that produced the surprise invitation to accompany the university baseball team to Japan, where it had been invited to play against the teams of Waseda and Keio Universities. The University granted seven months leave, and he went with the team as faculty representative, and returned by going around the world. The trip seems to have had no scientific connotations, but he retained a wealth of interesting memories.

In June of 1912 Bliss was married to Helen Hurd, who also was of a Kenwood family. They had two children, Elizabeth (Mrs. Russell Wiles), born in 1914, and Gilbert Ames, Jr., born in 1918.

In 1913 Bliss was promoted to a full professorship. When the United States entered the First World War, the Department of Mathematics undertook to teach navigation to men about to enter the naval school in Chicago. Bliss taught about a hundred of these. But in the summer of 1918 Oswald Veblen began pressing him to come to Aberdeen Proving Ground to join the mathematicians there in their urgent task of devising mathematical methods adequate for modern artillery. He was reluctant to do this, feeling that his work at the University was his best contribution to the country. But eventually he was persuaded to go to Aberdeen. It was a good decision, for here he made a contribution of considerable and lasting importance, which will be described in a later paragraph. He was still busy writing this for permanent record in December, 1918, when he received a telegram that his wife had been stricken in the influenza epidemic. She survived his return only a few days.

Two years later he was married to Olive Hunter, also a native of Chicago. They had almost thirty-one years of a happy life together. From 1931 on they made their home in Flossmoor; there are many who will vividly remember the charming home and the gracious and hospitable host and hostess.

In 1927 E. H. Moore was past the normal age of retirement, and wished to resign as head of the department. Bliss was appointed chairman, at first without public announcement. From the end of the war until his retirement Moore had tried to hold down the size of the permanent staff in the face of unprecedentedly large enrollments. Bliss took over the chairmanship when the enrollments had begun a slight decline which with the great depression turned into a rapid shrinkage. Enlarging the department was out of the question; but because Moore had held the size down, and because Bliss was always willing to exert himself to the utmost for his staff, there were no dismissals and no reductions in salary.

One important and conspicuous memorial to Bliss's chairmanship is Eckhart Hall at the University of Chicago. The President of the University, Max Mason, had obtained funds for the new building from Bernard Albert Eckhart, Julius Rosenwald, and the Rockefeller Foundation. Bliss gave unstintingly of his time and energy in the planning of the building and its furnishings—I recall a session of testing chairs for comfort, and a mathematical conference with Bliss that had to be abandoned after a few minutes because he was exhausted—but the results justify the expenditure. However, there is another memorial to his chairmanship in the large group of Chicago students of mathematics spread around the country. Mason repeatedly exerted pressure on the department to limit its student body by tightening requirements for admission. Bliss opposed this as a matter of principle. He felt that sometimes brilliant beginners fade away, while others develop real power after a weak start. Moreover, many who have not the capacity to take an advanced degree become better teachers because of studying advanced courses. This does not at all mean that Bliss condoned cheapening of the higher degrees. He felt that the time to select carefully among students is at the time of application for candidacy for a higher degree: "The candidates for higher degrees are the ones who take the time, not the listeners in lecture courses."

Concerning the doctorate, I quote his own words: "It seems to me that there is wide-spread misunderstanding of the significance of doctor's degrees in mathematics. The comment is often made that the purpose of such a degree is to train students for research in mathematics, and that the success of the degree is doubtful because most of those who obtain it do not afterward do mathematical research. My own feeling about our higher degrees is quite different. The real purpose of graduate work in mathematics, or any other subject, is to train the student to recognize what men call the truth, and to give him what is usually his first experience in searching out the truth in some special field and recording his impressions. Such a training is invaluable for teaching, or business, or whatever activity may claim the student's future interest."

In 1933 Bliss was appointed to the Martin A. Ryerson Distinguished Service Professorship, retaining the headship of the department. In 1935 the University of Wisconsin awarded him the honorary degree of Doctor of Science. In 1941, being sixty-five years old, he retired, but did not abandon mathematical activity; his book *Mathematics for Exterior Ballistics* was published during the war, in 1944, and his great *Lectures on the Calculus of Variations* in 1946. However, his health declined slowly during the next several years, and rapidly in 1951. He died in Flossmoor on May 8, 1951, one day before his seventy-fifth birthday.

Occasionally one hears from a Ph.D. some bitter remark about the man under whom he wrote a dissertation. I have never heard such a remark about Bliss, and am confident that none was ever made. He was by nature kind, and his criticisms of students' work never stung. He enjoyed teasing his younger colleagues, always with an air of innocence, and never maliciously. As a small example, shortly after he became head of the Police Commission of Flossmoor he solemnly handed several guests a set of catalogues of police insignia and had them search out the largest and most decorative police star listed, quite as seriously as though he really intended to wear one.

Bliss had many mathematical activities besides those within the department. In 1916 he was elected to the National Academy of Sciences. He also became a member of the American Philosophical Society (1926) and a fellow of the American Academy of Arts and Sciences (1935). He was president of the American Mathematical Society in 1921 and 1922, when its financial situation was difficult. With E. R. Hedrick he devoted a great deal of time and energy to a campaign to increase membership, resulting in an increase of about fifty percent. In 1930 he was Vice President and Chairman of Section A of the American Association for the Advancement of Science. He was also a member of the Mathematical Association of America, the Illinois Academy of Science, the London Mathematical Society, the Deutsche Mathematische Verein, and the Circolo Matematico di Palermo. He was associate editor of the *Annals of Mathematics* from 1906 to 1908, and of the *Transactions of the American Mathematical Society* from 1909 to 1916. For many years he was an editor of the "Carus Mathematical Monographs," and (after 1929) chairman of the editorial committee of the "University of Chicago Science Series." For twelve years (1924–1936) he served on the Fellowship Board of the National Research Council. This he did conscientiously, although he found no pleasure in ranking people according to estimated ability; this was against his natural inclination to find what was good in each man and to encourage it. Nevertheless, a statistical self-evaluation of the Board indicated clearly that its work had been well done.

There were other activities, too. For some years he was a trustee of the Teachers Insurance and Annuity Association. In Flossmoor he served as a member of the Village Board of Trustees and as head of the Police Commission. His ability to speak clearly and interestingly caused him to be invited often to speak informally on scientific subjects.

Looking through his list of publications, one is struck by the way in which the calculus of variations serves as center of attraction. There are departures, but always there is a return. Moreover, a few details of his way of thinking are clear. He must have visualized clearly; a curve was a picture in his mind, not a system of functions. He did not seek maximal generality, but preferred to exhibit the problem's true center of interest with clarity. Thus he did not choose to use Lebesgue integrals in the calculus of variations, presumably feeling that the real interest lay in the behavior of families of smooth curves, and that the extension of the theory, for example to all curves of finite length, could be added afterwards if one wished. He had no tendency to join any "Pythagorean brotherhood"; he wrote carefully so as to be easily intelligible to as many readers as possible. It was most appropriate that when the Chauvenet Prize for mathematical exposition was first awarded by the Mathematical Association of America, in 1925, Bliss was the recipient. (The paper for which he received the award was "Algebraic Functions and their Divisors," listed in the bibliography as D43.)

His doctoral dissertation (D3) concerned geodesics on a torus. Points on a torus, or anchor ring, can be located by means of functions which depend on two angles and are thus doubly periodic. Bliss found the specific formulas for the geodesics in terms of elliptic functions. He thus could show that through every point there pass geodesics which cross the inner equator, all such geodesics being free of pairs of conjugate points; and through every point not on the inner equator there pass geodesics which do not cross that equator, and on such geodesics each point has conjugate points. This paper is an addition to the small collection of interesting special problems whose detailed discussion is the ground-stratum for generalizations.

Papers D4 (written in Minnesota) and D5 (written in Göttingen) can be thought of as a pair. In a plane, the shortest curve joining a fixed point A to a fixed curve C is a line-segment which is perpendicular to C at the point of intersection. There is however another condition; the center of curvature of C at B must not lie between A and B. If we strengthen the condition by demanding that A itself is not the center of curvature, the conditions are sufficient to guarantee that the segment has minimum length when compared with nearby curves from A to C. Bliss extended this to general plane problems in D4, and in D5 gave the first complete treatment of the case in which both end points are permitted to vary along fixed curves.

Two fundamentally important elements of differential geometry are the expression for the length of a curve and the idea of geodesic. But the length is given by an integral of the kind studied in the calculus of variations, and the extremals of this integral are the geodesics. Bliss, being interested in geometry and in the calculus of variations, wondered if the calculus of variations could also furnish generalizations of other parts of differential geometry. He published three papers (D11, 14, 27) concerned with extremals in two-dimensional space, and obtained partial generalizations of several theorems. These papers now would be said to be on "two-dimensional Finsler geometry." Finsler treated the n-dimensional case, but not until 1918. Although Finsler geometry is partly swallowed up by tensor analysis, it still retains a measure of independent existence and is still being studied.

Because the calculus of variations deals with families of curves satisfying the Euler-Lagrange differential equations and satisfying some sort of end-condition, it has use for existence theorems of considerable strength. Bliss found the need of these theorems and of theorems on implicit functions. He wrote several papers (D6, 9, 20, 33) and gave a systematic presentation of results in the Princeton Colloquium lectures (A1). It is characteristic of Bliss's work that the theory is developed in all the generality called for by the applications; that the generalization into the realm of the Lebesgue methods was eschewed, presumably because the applications did not call for it; and that the methods invented and the style of writing made the subject easily and pleasantly accessible.

The implicit functions theorem has, as one of its applications in the calculus of variations, the use of determining the parameters which select from a given family of curves (extremals) that particular curve which passes through a given point. Under some conditions, for example, when all the extremals are tangent to one fixed curve, the problem requires solving a system of equations near a singular point. If the functions involved are (real) analytic, the "preparation theorem" of Weierstrass is an important tool. Bliss extended this to the generality needed, and gave an elegant demonstration (D18, 21).

From the study of singular points of analytic transformations to the study of algebraic curves is a natural step. Bliss, having made significant contributions to the one, became interested in the other (D41, D42, D43, A4). Since he liked to think in geometric images, he of course studied the curves from the geometric and analytic point of view. The more recent developments have shown that algebraic methods yield strong results in this field. Nevertheless, Bliss's work on the geometric theory left that aspect of the subject in a much more complete and coherent state than it had previously attained.

Measured by utilitarian standards there can be no doubt that Bliss's outstanding contribution was in ballistics; and this contribution is by no means trivial even from a pure mathematical point of view. A modern firing table has two essential parts. Both refer to a given combination of gun, projectile, and charge. One gives the elevation necessary to attain a desired range under "normal conditions"; the other gives the change in this elevation needed to correct for the way in which conditions at time of firing differ from standard. These include effects of wind, of non-standard density, of non-standard projectile-weight, and others. At the beginning of the First World War all the nations involved were computing trajectories by the method of Siacci, based on approximations adequate for projectile traveling in level, flat trajectories. But guns were being used at long ranges and high elevations, and the Siacci theory was no longer adequate. "Fudge-factors" were introduced to make *ad hoc* corrections, but these were neither theoretically sound nor usefully accurate. F. R. Moulton replaced these outgrown devices by a method of numerical integration similar to that used in computing orbits, and capable of very high accuracy.

This took care of the problem of the trajectories under "normal conditions" so well that it continues to be used, unchanged except in detail, even with the electronic computers of today. However, Moulton's treatment of the corrections was less satisfactory both theoretically and computationally. The change of range due to wind (or to non-standard density, or temperature) is an example of a "functional"; it is a function which is not determined by a single number, the "independent variable," nor indeed by any finite set of numbers, but is determined only when we know the entire course of another function, namely, the wind at each altitude. When the Siacci method was standard, there was no serious attempt to discuss this difficult dependence; some crude over-simplification was used, such as replacing the actual variable wind by a constant, agreeing with the actual wind at two thirds of the maximum altitude reached by the projectile. It was possible to devise methods applicable to the trajectories computed by Moulton's numerical integration process, but these methods needed both mathematical justification and computational simplification.

Bliss's work in analysis had provided him with just the appropriate background for this problem. He had been led to study functionals, and published one paper (D28) on the subject, and had guided the writing of three doctoral dissertations (C. A. Fischer, 1912; Miss Le Sturgeon, 1917; I. A. Barnett, 1918) on the subject. Thus he could contribute quickly to the work of the group, and soon provided both the mathematical foundation and the computational procedure.

To handle the mathematical foundation, he devised a numerical measure, or norm, to specify the "size" of a disturbance; and he proved that the effect of a disturbance can be approximated by a linear estimate close enough so that for small disturbances, the error is an arbitrarily small percent of the norm of the disturbance, no matter what the shape of the disturbance (e.g., the pattern of winds) might be (D39, D40, A5). This linear estimate is the "differential effect" of the disturbance, and is exactly what is used in service. It is not difficult to find the system of differential equations which the differential effect must satisfy; in fact, these were known before Bliss gave the proof that the differential effect really exists.

The practical side of the problem was to devise a method of handling these equations so that given a variety of disturbances, the range-effect of each disturbance could be quickly computed. Bliss did this (D35, 36, 37) by introducing another system of differential equations related to the equations for the differential effect, and called the "adjoint system." For each undisturbed trajectory a single solution of the adjoint system is found, determined by a certain known set of values at the end of the trajectory. This one solution brings us almost to the end of the problem. For, given any specific disturbance function, we need only multiply it into the solution of the adjoint system and perform a numerical quadrature (say, by Simpson's rule) to obtain the range effect of the disturbance. As compared with the best of preceding methods, this device saves about three quarters of the work. It continued in use, with at most small amendments, through all of the Second World War. The advent of high-speed computing machines took away some of its pre-eminence, but it is not likely that such a convenient method will be permanently shelved; in some modification, it will probably be an auxiliary in any computing program involving effects of small changes in the data.

After his war work, Bliss returned to analysis, centered again on the calculus of variations. He did not contribute to the new theory begun by Morse, nor to the direct-methods theory of Tonelli (although he encouraged at least one student to work in that field). The inverse problem received some attention from him; in 1908 he had published a paper (D15) on the subject, and he directed three graduate students (D. R. Davis, L. La Paz, N. A. Moscovitch) in writing dissertations on this problem. Likewise, he gave some attention to multiple integral problems. Although he published only one paper on the subject (D59), he conducted several courses and seminars on the subject, and published mimeographed notes.

However, the center of his research was consistently the single-integral problem of the calculus of variations. Of the various forms of such problems he chose the problem of Bolza, since it most readily specialized down to include the other forms. The result of his work is his fine book *Lectures on the Calculus of Variations* (A6), published in 1946, after his retirement. Besides his own work this contains the advances made by other workers in the field. But so much of this other work was done by men who had studied under Bliss and received their inspiration from him that the book is to an unusual degree a monument to Bliss himself.

If it were necessary to pick out one of his contributions as a special exhibit, the choice almost certainly should be his treatment of the Jacobi condition (D30). This had been handled, since the middle of the nineteenth century, by an analytic device called the transformation of the second variation; but for the simplest problems this was cumbersome, and for the more complicated problems it was hopelessly unwieldy. A geometric substitute due to Kneser was elegant for the simplest problems, but did not seem to extend to more involved ones. Bliss remarked that the second variation of an integral is itself given by an integral of the same type as we started with, and if it is never negative then the identically-vanishing variation gives it its least value, 0. From this the desired results follow quite readily. Moreover, the device is equally applicable to the more complicated problems, such as the Lagrange and Bolza problems. The principal purpose of the transformation of the second variation is attained without the transformation. But even the secondary results of the transformation are not lost in the process. For by use of the Bliss technique the second variation can be transformed into the form desired by earlier investigators, without any of the cumbersome analytic machinery of previous methods. As an indication of how "right" Bliss's method is, less than four years after its publication it was referred to, in a paper published by another mathematician, as the "classical" method.

This was one of his many contributions to what might be called a program of saving the calculus of variations from death by elephantiasis. The life of a mathematical science comes from its intellectual attractiveness. In the past it has happened that some branch of mathematics has become bulky by the piling up of minutiae and the long-winded discussion of intricate and often uninteresting problems by methods stretched out beyond their domain of appropriateness. Such branches naturally lose all appeal, and become senile unless rejuvenated by new ideas and re-thinking that succeed in attaining the principal results (and new ones, too) more readily and more beautifully. In the early twentieth century the calculus of variations was in danger of losing its appeal because of mounting complexity. How much Bliss contributed to its rescue, as well as to its advancement, can be seen by anyone who will compare the compactness and generality of the theory in the *Lectures on the Calculus of Variations* with the older papers on the same subject. It is a worthy monument.

KEY TO ABBREVIATIONS

Amer. Jour. Math. = American Journal of Mathematics
Amer. Math. Mo. = American Mathematical Monthly
Ann. Math. = Annals of Mathematics
Bull. Amer. Math. Soc. = Bulletin of the American Mathematical Society
Jour. London Math. Soc. = Journal of the London Mathematical Society
Jour. U. S. Artil. = Journal of United States Artillery
Math. Ann. = Mathematische Annalen
Pop. Astron. = Popular Astronomy
Proc. Nat. Acad. Sci. = Proceedings of the National Academy of Sciences
Trans. Amer. Math. Soc. = Transactions of the American Mathematical Society

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