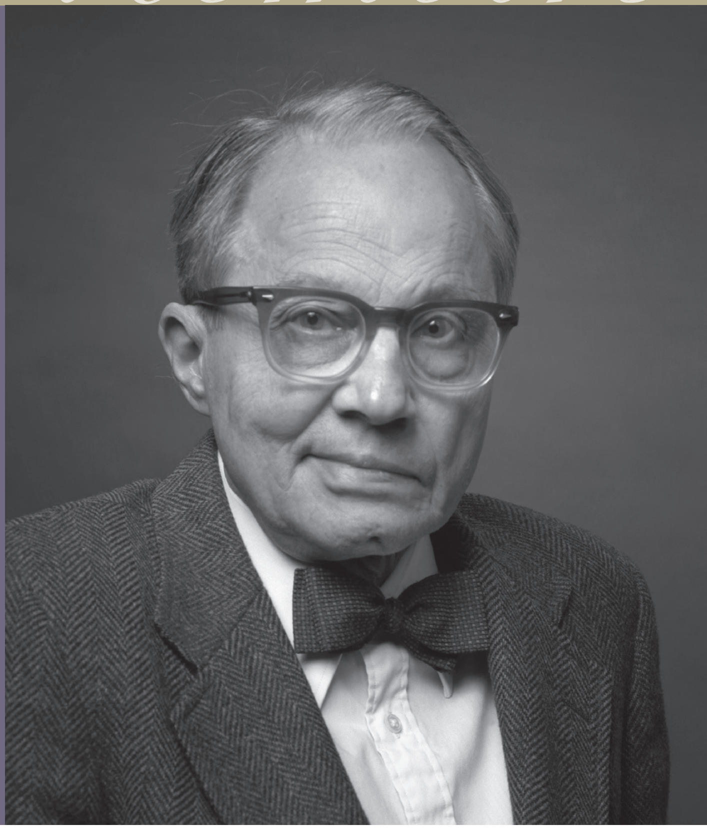
Frederick W. Gehring

1925–2012

BIOGRAPHICAL *Memoirs*



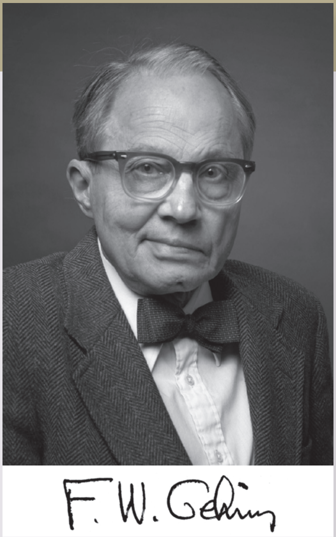
A Biographical Memoir by
Gaven J. Martin

NATIONAL ACADEMY OF SCIENCES

FREDERICK WILLIAM **GEHRING**

August 7, 1925–May 29, 2012

Elected to the NAS, 1989



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By Gaven J. Martin

Frederick W. Gehring was a hugely influential mathematician who spent most of his career at the University of Michigan (UM). Appointed to the faculty in 1955, he was the T. H. Hildebrandt Distinguished University Professor from 1987 on. Gehring's major research contributions were to geometric function theory, particularly in higher dimensions $\mathbb{R}^n$, where $n \geq 3$. Gehring developed this field in close coordination with colleagues, primarily in Finland, over the three decades from 1960 to 1990. His seminal work in geometric function theory—notably, by making important connections with geometry and nonlinear partial differential equations to solve major problems—helped drive the field forward.

Over the course of his career, Gehring received numerous honors from the international mathematical community. He was invited three times to address the International Congress of Mathematicians—in Moscow in 1966, Vancouver in 1974, and Berkeley (a plenary lecture) in 1986—and was awarded honorary degrees from Cambridge University (1976), from Finland's University of Helsinki (1979) and University of Jyväskylä (1990), and from the Norwegian University of Science and Technology (1997). In 1989, Gehring was elected to the American Academy of Arts and Sciences and the U.S. National Academy of Sciences.

Other honors included an Alexander von Humboldt Foundation fellowship (1981–84), an Order of Finland White Rose, Commander class (Finland's highest scientific honor for foreigners), and a Lars Onsager Professorship at Norway's University of Trondheim (1995–1996). Gehring served for 19 years on various committees of the American Mathematical Society, and in 2006 he received its Steele Prize for Lifetime Achievement. He also served three terms as chairman of the University of Michigan's Department of Mathematics, and played a leading role in shaping it through the latter part of the 20th century.

Gehring was born in Ann Arbor, MI, to a family of German origin. His great-grandfather Karl Ernst Gehring (1829–1893) had emigrated from Germany in 1847 and settled in Cleveland, OH, where he founded the Gehring Brewery. Frederick's grandfather Frederick William Gehring (1859–1925) was treasurer of the brewery and cofounder of a Cleveland bank. Frederick's father Carl Ernest Gehring (1897–1966) loved music and was an amateur composer. He came to Ann Arbor to study engineering at UM but soon switched to journalism and later worked for the *Ann Arbor News* as state news editor and music critic. Frederick's mother Hester Reed Gehring (1898–1972) was the daughter of John Oren Reed (1856–1916), a physics professor at UM who later became a dean.

Frederick grew up in Ann Arbor and graduated from University High School in 1943. He was then accepted by the Massachusetts Institute of Technology to study physics or engineering, but he chose instead to enlist in the U.S. Navy V-12 program. By coincidence, the Navy sent him to Ann Arbor for a special program in electrical engineering. He later graduated with a double major in electrical engineering and mathematics. By that time the war was over, but the Navy sent him to sea for four months.

Upon his return, Gehring re-enrolled at UM and decided, at the suggestion of math professor Ruel Churchill, to concentrate on mathematics. After receiving an M.A. degree from UM in 1949, Gehring attended the University of Cambridge on a Fulbright Scholarship. He earned his Ph.D. in 1952, having taken courses from such famous mathematicians as J. E. Littlewood and A. S. Besicovitch while writing a thesis under the direction of J. C. Burkill.

At Cambridge, Gehring met Lois Bigger, who had come from Iowa, also on a Fulbright Scholarship, to continue her study of microbiology. She too received her Ph.D. from Cambridge in 1952. When they returned to the United States, they settled not far apart—Gehring went to Harvard University as a Benjamin Peirce Instructor and Bigger went to Yale University on a research fellowship—and continued to see each other. They were married on August 29, 1953, in Bigger's hometown of Mt. Vernon, IA.

During Gehring's time at Harvard, his acquaintance with mathematics department colleague Lars Ahlfors drew him closer to the area of complex analysis. When his position at Harvard ended in 1955, Gehring joined the mathematics faculty of UM, where he remained based for the rest of his life. But a turning point in Gehring's mathematical career, involving two years spent in Europe, soon began, in 1958. His colleague Jack Lohwater had spent the year 1956–57 in Helsinki, Finland, where he saw that Olli Lehto and Kalle Virtanen, among others, were working on the theory of normal functions. Encouraged also by Ahlfors, Gehring decided to go to Helsinki too. Lohwater put him in touch with Lehto, and Gehring arranged to spend the year 1958–59 there.

But when he arrived, Gehring learned to his dismay that his Finnish hosts were no longer interested in meromorphic functions; they were now working on quasiconformal mappings. Gehring's reaction, as Lehto later recalled, was, "Fine, I like quasiconformal mappings – what are they?" He was told he could soon learn, as they were about to start a seminar on the topic, in Finnish! Gehring responded to the challenge by learning, in a hurry, to speak Finnish.

In the year 1959–60, Gehring moved to Zürich, where Albert Pfluger had also been working on quasiconformal mappings. Stimulated by discussions with Pfluger and a paper by Charles Loewner, Gehring began to further develop the higher-dimensional theory of quasiconformal mappings. In those two years abroad, Gehring made professional contacts that would strongly influence the entire course of his career. When he returned to Michigan in 1960, he began training students in quasiconformal mappings, and his first graduated from UM in 1963. During his career, Gehring would direct 29 Ph.D. students in all, most of whom would go onto active careers in teaching and research at academic institutions.

An important aspect of Gehring's professional work was his extensive service as an editor. He was on the editorial boards of nine research journals and was a book-series editor for Van Nostrand (1963–70), for North Holland (1970–94), and most famously for Springer-Verlag (1974–2003), notably with its Undergraduate Texts in Mathematics series.

At the University of Michigan, Gehring was promoted to professor in 1962, was named to a collegiate chair in 1984, and became the T. H. Hildebrandt Distinguished University Professor in 1987. He served three terms as chairman of the Department of Mathematics, in 1973–75, 1977–80, and 1981–84. The University honored Gehring with a Distinguished Faculty Achievement Award in 1981, the Henry Russel Lectureship in 1990, and a Sokol Faculty Award in 1994. An international conference on "Quasiconformal Mappings and Analysis" was held in Ann Arbor in August 1995 on the occasion of his 70th birthday. He retired in 1996.

In 2006, the American Mathematical Society honored Gehring with its Steele Prize for Lifetime Achievement. "Largely because of Gehring's work," the accompanying citation stated, "the theory of quasiconformal mappings has influenced many other parts of mathematics, including complex dynamics, function theory, partial differential equations, and topology. Higher-dimensional quasiconformality is an essential ingredient of the Mostow rigidity theorem and of recent work of Donaldson and Sullivan on gauge theory and four-manifolds.…Gehring's mathematics is characterized by its elegance and simplicity and by its emphasis on deceptively elementary questions [that] later become surprisingly significant."

Gehring died in Ann Arbor on May 29, 2012 at age 86, after a long illness.

Gehring's mathematics

Gehring had wide and varied interests: real and complex analysis, geometric function theory, partial differential equations (PDE), discrete groups and hyperbolic geometry, and geometric group theory, just to name a few. He collaborated widely, with nearly 40 different coauthors, and wrote or co-wrote some 130 mathematical papers, starting with his first article in 1951. Gehring's Ph.D. thesis at Cambridge was ostensibly under the supervision of J. C. Burkill, but he felt that J. E. Littlewood and A. S. Besicovitch were equally involved in his research directions and mentorship. Indeed, he thanked Besicovitch for the problem that gave rise to his first published paper.

Gehring's paper *Quasiconformal mappings in Euclidean spaces* (Gehring 2005) gave not only a fairly concise summary of his perspective of the main results achieved since the 1930s in the theory of quasiconformal mappings but also imparted his broad vision of how they connect with other areas of mathematics. Gehring had certainly been a leader in many of these developments.

Early years

Gehring's earliest significant paper, *On the total differentiability of functions of a complex variable*, was co-written with Lehto during the American's first visit to Finland (Gehring and Lehto 1959). That paper is still a classic, and is often used by researchers. The result is false in higher dimensions, but it is the interplay between topology and analysis in the proof that was one of Gehring's favorite achievements. The main theorem of the paper is the following:

Theorem 1 Let $f:\Omega\rightarrow\mathbb{C}$ *be a continuous open mapping of a planar domain* Ω *. Then* f *is differentiable almost everywhere in* Ω *if and only if* f *has finite first partials almost everywhere.*

Gehring, with, from left, Jussi Väisälä, Lalevi Suominen, and Olli Lehto.

D. Menchoff had obtained this result earlier, in 1931, for planar homeomorphisms. But a consequence of Gehring and Lehto' theorem, which came to underpin much of planar geometric function theory, is that every discrete open Sobolev mapping $f\in W^{1,1}_{loc}(\Omega,\mathbb{C})$ is differentiable almost everywhere. Using this result one can connect the volume derivatives and the pointwise Jacobian of a mapping and thereby obtain a first version of the change-of-variables formula. For solutions to PDE capacity estimates and a maximum principle typically give openness and discreteness, and so the result has wide application in the theory of two-dimensional nonlinear PDE and the topological properties of their solutions.

Higher-dimensional quasiconformal mappings

In early 1960, Gehring began the task of laying down some of the now-basic tools of quasiconformal mappings in higher dimensions. Others were working too in this area, most notably Jussi Väisälä (in Finland) and Yu. G. Reshetnyak (in Russia). Gehring wrote a long review of Väisälä's papers, *On Quasiconformal Mappings in Space* and *On Quasiconformal Mappings of a Ball.* The major tool in Väisälä's work was the extremal length-method, about which Gehring and Väisälä had already co-written a paper, "On the Geometric Definition for Quasiconformal Mappings" (Gehring and Väisälä 1961). The two authors' ideas for planar quasiconformal mappings built on the work of Pfluger, Ahlfors, and Arne Beurling in 1946.

Initially, Gehring and Väisälä were seeking to show the equivalence of the various natural generalizations to higher dimensions of (the many equivalent definitions for) planar quasiconformal mappings, given below. They also sought extensions of other well-known planar results to higher dimensions.

Here then are two apparently independent definitions for a quasiconformal mapping. First the *analytic definition.* Suppose f is a homeomorphism that lies in the Sobolev space $W_{loc}^{1,n}(\Omega,\mathbb{R}^n)$ of functions whose first derivatives are integrable with exponent $n =$ dimension. Then f is K*-quasiconformal*, $1 \leq K < \infty$, if

$$|Df(x)|^n \leq K\ J(x,f), \text{ for almost all } x \in \Omega. \tag{1}$$

Here Df is the $n \times n$ matrix differential of f, $J(x,f)$ is its determinant (Jacobian), and $|\cdot|$ is the operator norm.

Second, a *geometric definition*. Let Γ be a family of curves in Ω—for instance, the family of all curves connecting two sets $E,F \subset \Omega$. A non-negative Borel integrable function $\rho : \Omega \to \mathbb{R}_{\geq 0}$ is admissible for Γ if for each $\gamma \in \Gamma$ we have

$$\int_\gamma \rho(s)\, ds \geq 1$$

The modulus of Γ is

$$M(\Gamma) = \inf\{\int_\Omega \rho^n(x)\, dx : \rho \text{ is admissible for } \Gamma\}$$

We say that a homeomorphism $f : \Omega \to f(\Omega)$ is $\tilde{K}$ *-quasiconformal,* $1 \leq \tilde{K} < \infty$, if for every curve family $\Gamma \subset \Omega$

$$\frac{1}{\tilde{K}} M(\Gamma) \leq M(f\Gamma) \leq \tilde{K}\, M(\Gamma) \tag{2}$$

Here $f\Gamma = \{f \circ \gamma : \gamma \in \Gamma\}$ is a curve family in $f(\Omega)$.

Hadamard's inequality gives $|Df(x)|^n \geq J(x,f)$, and so equality holds in (1) for $K=1$. If also the dimension $n=2$, a little calculation reveals the Cauchy-Riemann equations. Further, the Looman-Menchoff theorem implies the weaker hypothesis that $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$ would suffice to conclude f is conformal. For (2), when $\tilde{K}=1$, the argument is a little longer to deduce that f is conformal, but the relationship to the length-area inequality for conformal mappings is clear. In dimension $n \geq 3$, the constants K and $\tilde{K}$ may differ for the same mapping, but both are simultaneously equal to 1 (easy) and simultaneously finite or infinite (a theorem). Thus the space of quasiconformal mappings is well defined, but when specifying constants one must refer to a particular definition.

Gehring's two major papers in this area—"Symmetrization of Rings in Space" (Gehring 1961) and "Rings and Quasiconformal Mappings in Space" (Gehring 1962—both appeared in *Transactions of the American Mathematical Society,* after an announcement of the main results was published earlier in 1961 in *Proceedings of the National Academy of Sciences* (47:98–105).

In these works, Gehring started with results on the spherical symmetrization of rings in space, which were motivated by, though quite different from, earlier analyses of the two-dimensional case by G. Bol, T. Carleman, G. Pólya, and G. Szegö, among others. Gehring's symmetrization basically identified extremal configurations for rings (read conformal invariants) in terms of geometric information such as diameters of components and distances between them, which generalized the "length-area" method from complex analysis. The resulting estimates on arbitrary rings led to direct proofs of such things as the Hölder continuity of higher-dimensional quasiconformal mappings (previously established by E. D. Callender, based on arguments of C. B. Morrey and L. Nirenberg). Naturally, this modulus of continuity would give compactness via the Arzela-Ascoli theorem. Next, Gehring proved the existence and uniqueness of an extremal function realizing the Loewner capacity of a ring (doubly connected domain) in space. This extremal function u solves the nonlinear PDE $div(|\nabla u|\nabla u)=0$. Gehring used these results to establish equivalences between the analytic definition and the geometric definition.

The most recognizable thing that Gehring proved with these two papers was a sharp form of the Liouville theorem, which dated from 1850, when Joseph Liouville added a short note to a new edition of Gaspard Monge's classic work, *Application de l'Analyse à la Géometrie*, whose publication Liouville was overseeing. This note established strong rigidity for higher-dimensional conformal mappings.

Liouville proved that if $f:\Omega \to \mathbb{R}^3$ is a 3-times continuously differentiable conformal mapping, then there is a Möbius transformation $\Phi:\overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ such that $\Phi|\Omega = f$. This is a very surprising rigidity theorem, as in two dimensions the space of smooth conformal mappings is infinite-dimensional and there is no local-to-global extension theorem. One significant corollary of Liouville's theorem is that the only subdomains Ω in $\mathbb{R}^n$ with $n \geq 3$ that are conformally equivalent to the unit ball $\mathbb{B}^n$ are Euclidean balls and half-spaces. This stands in stark contrast to the famous Riemann mapping theorem, announced in 1851 a year after Liouville's note was published: Any simply connected proper subdomain $\Omega \subset \mathbb{C}$ of the complex plane is conformally equivalent to the unit disk $\mathbb{D}$.

Certainly, conformal mappings are 1-quasiconformal mappings. What of the converse? Gehring had by now developed the tools to identify extremal rings and their configurations, and he had gained an understanding of how moduli behaved under continuous deformations and how this gave regularity for quasiconformal mappings. It was a short step (but the one he had been aiming for) to prove the following theorem:

Theorem 2 *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 3$, *and let* $f:\Omega \to f(\Omega)$ *be a 1-quasiconformal mapping. Then* f *is the restriction to* Ω *of a Möbius transformation of* $\overline{\mathbb{R}}^n$.

Gehring with Lars V. Ahlfors.

Gehring lecturing on QC-Schönflies Theorem.

There are now refinements of this result, and the curious discrepancy between what is known in even and odd dimensions remains one of the most interesting problems in the higher-dimensional theory.

Ahlfors's review of these two papers of Gehring began, "This is an announcement of important results in the theory of 3-dimensional quasiconformal mappings," and it ends, "In a final section [Gehring] proves that a 1-quasiconformal mapping is a Möbius transformation. This is Liouville's theorem without any regularity assumptions—a remarkable achievement."

This last note was important to Gehring, as Ahlfors was a personal friend and mathematical hero to him, perhaps above all others.

Area distortion

Until Kari Astala solved it (Astala 1994), the most famous open problem about planar quasiconformal mappings was the area distortion conjecture raised in the paper "Area Distortion Under Quasiconformal Mappings" (Gehring and Reich 1966). Suppose that $f(\mathbb{D}) = \mathbb{D}$ and $f(0) = 0$, and given $E \subset \mathbb{D}$, its measure is denoted by $|E|$. Bojaski's regularity result for planar quasiconformal mappings shows that for each $K \geq 1$ there are numbers a_K and b_K such that if f is K-quasiconformal, then

$$\frac{|f(E)|}{\pi} \leq b_k \left(\frac{|E|}{\pi} \right)^{a_k}, \text{ for each measurable set } E \subset \mathbb{D} \tag{3}$$

Gehring and Reich proved that $a(K) = K^{-a}$ for some $1 \leq a \leq 40$, and that $b(K) = 1 + 0(K-1)$ as $K \to 1$. They deduced this as a consequence of the inequality for the Beurling transform of characteristic functions, which they achieved using the Calderón-Zygmund theory of singular integral operators. They conjectured that $a = 1$, and Astala showed the area distortion conjecture is true. The other problem suggested by the Gehring and Reich paper concerned the L^p-norms of the Beurling transform, which still stands as one of the most challenging problems in the modern theory.

Higher integrability

We now come to the result that was undoubtedly Gehring's most well-known contribution to mathematics. Though the results have been improved, many of their underlying ideas are still used in mathematics today.

In “The L^p-Integrability of the Partial Derivatives of a Quasiconformal Mapping,” a remarkable 1973 paper that was actually a landmark in modern analysis and PDE, Gehring established that the Jacobian determinant of a K-quasiconformal mapping is integrable above the natural exponent (n = dimension). That is, the usual assumption $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, together with a bound as per (1), implies that $f \in W^{1,n+\varepsilon}_{loc}(\Omega, \mathbb{R}^n)$ for some $\varepsilon > 0$, depending on n and K (for which Gehring gave explicit estimates).

While this result was already known in the plane, due to the work of B. V. Bojarski (Bojarski 1957) and perhaps anticipated in higher dimensions, it is impossible to overstate how important this result has been in the theory of quasiconformal mappings and more generally in Sobolev spaces and in the theory of nonlinear PDEs. The techniques developed to solve this problem—the well-known reverse Hölder inequalities—are still among the main tools used in nonlinear potential theory, nonlinear elasticity, PDEs, and harmonic analysis. In the introduction to his 1973 paper, Gehring said the results follow, “using a quite-elementary proof.” But time has shown his analysis here to be deep and highly innovative. It is a mark of the clarity of Gehring’s writing (and his own modesty) that one can follow the arguments easily and overlook how hard these ideas are to come by.

We state the following version of Gehring’s 1973 result as proved in Iwaniec and Martin (2001), which also gave the result for quasiregular mappings:

Theorem 3 (Higher Integrability) *Let* $f : \Omega \to \mathbb{R}^n$ *be a mapping of Sobolev class* $W^{1,q}_{loc}(\Omega)$ *satisfying the differential inequality*

$$|Df(x)|^n \leq K\, J(x, f). \tag{4}$$

Then there are $\varepsilon_{K^*}, \varepsilon_K > 0$ such that if $q > n - \varepsilon_{K^*}$, then $f \in W^{1,p}_{loc}(\Omega)$ for all $p < n + \varepsilon_K$.

As an immediate corollary we have the following:

Theorem 4 Let $f : \Omega \to \mathbb{R}^n$ be a *K*-quasiconformal mapping. Then there is $p_K > n$ such that $f \in W^{1,p_K}_{loc}(\Omega)$.

The higher-dimensional integrability conjecture, perhaps the most important outstanding problem in the area, would assert that if f satisfies (4) and lies in $W^{1,q}_{loc}(\Omega)$

for some $q > nK/(K+1)$, then f actually lies in the Sobolev space $W^{1,p}_{loc}(\Omega)$ for all $p < nK/(K-1)$. In two-dimensions, this is the area distortion theorem. A key step in Gehring's proof was the following:

Lemma 1 (Reverse Hölder Inequality) Let $f : \Omega \subset \mathbb{R}^n \to \Omega'$ be a *K*-quasiconformal mapping. Then there is $p = p(n,K) > 1$ and $C = C(n,K)$ such that

$$\left(\frac{1}{|Q|} \int\int_Q J(x,f)^p \, dx \right)^{1/p} \leq \frac{C}{|Q|} \int\int_Q J(x,f) dx \qquad (5)$$

for all cubes Q such that $2Q \subset \Omega$.

This result is often referred to as the Gehring Lemma. Tadeusz Iwaniec's survey on this lemma contains generalizations and some of its far-reaching applications in mathematical analysis (Iwaniec 1998).

Next, one deduces from the improved regularity bounds on the distortion of Hausdorff dimension:

Theorem 5 If $f : \to \mathbb{R}^n$ is *K*-quasiconformal, then there is $a \geq 1$ such that for each $E \subset \mathbb{R}^n$,

$$K^{-a} \left(\frac{1}{\dim_H(f(E))} - \frac{1}{n} \right) \leq \frac{1}{\dim_H(f(E))} - \frac{1}{n} \leq K^a \left(\frac{1}{\dim_H(f(E))} - \frac{1}{n} \right)$$

where $\dim_H$ refers to the Hausdorff dimension.

This result shows in particular that sets of 0- and *n*-dimensional Hausdorff measure are preserved, proving an earlier conjecture that Gehring made in his work with Väisälä. The optimal conjecture would be that $a = 1$ here.

Gehring always gave credit to E. De Giorgi's paper (De Giorgi 1957) to support his proof. While De Giorgi's work (and also that of J. F. Nash) was hugely important, it occurred substantially earlier and was well known in the PDE community by the time of Gehring, who provided considerable insight and applied new ideas to the problems in this area.

Discrete groups and hyperbolic geometry

In the last couple of decades of Gehring's research career, he and the author worked on the geometry of discrete groups and jointly wrote some 30 papers on the subject. Motivated by his attendance at Alan Beardon's series of lectures on the geometry of discrete groups in Ann Arbor around 1980, this represented a new research direction for Gehring.

He started by thinking about how the general theory of Möbius groups would generalize to discrete groups of quasiconformal mappings—typically, with an assumed uniform bound on the distortion; to a surprising extent, the general theory used only the compactness properties of quasiconformal mappings. These initial studies were documented in the paper "Discrete quasiconformal groups. I" (Gehring and Martin 1987). The theory of convergence groups developed in that work was a timely idea that meshed well with Mikhael Gromov's theory of hyperbolic groups published in the same year. In the hands of Pekka Tukia, Gehring and Martin's results were quickly used to characterize certain conjugates of Möbius groups among groups of homeomorphisms (Tukia 1988)—a program that was completed by David Gabai, Andrew Casson, and Douglas Jungreis, and yielding a far-reaching generalization of the Nielsen Realization problem.

These efforts, together with earlier work by Geoffery Mess, led directly to the resolution of the Seifert conjecture: a compact orientable irreducible 3-manifold with a fundamental group having an infinite-cyclic normal subgroup is Seifert-filtered. And Mike Freedman even determined an equivalence between the four-dimensional surgery conjecture and an extension problem for convergence groups of $\overline{\mathbb{R}}^3$. These new and surprising connections between quasiconformal mappings and low-dimensional topology were enormously exciting to Gehring.

A group of homeomorphisms acting on $\overline{\mathbb{R}}^n$ is called a *quasiconformal group* if there is some finite K such that each $g \in G$ is K-quasiconformal. In two dimensions, every quasiconformal group acting on $\overline{\mathbb{R}}^2 = \hat{\mathbb{C}}$, the Riemann sphere, is the quasiconformal conjugate of a conformal group or a Möbius group—a result of Dennis Sullivan. Thus there is a quasiconformal $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and a group of Möbius transformations Γ such that $G = f\Gamma f^{-1}$. On the other hand, if $n \geq 3$, then there is a discrete quasiconformal group not quasiconformally conjugate to a Möbius group.

A convergence group G is a group of self-homeomorphisms of $\overline{\mathbb{R}}^n$ that has the following property:

Every infinite subfamily of G contains a sequence $\{g_j\}$ such that one of the following holds:

1. There is a self-homeomorphism g of $\overline{\mathbb{R}}^n$ such that $g_j \to g$ and $g_j^{-1} \to g^{-1}$ uniformly in $\overline{\mathbb{R}}^n$ as $j \to \infty$.

2. There are points x_0 and y_0 in $\overline{\mathbb{R}}^n$ such that $g_j \to y_0$ and $g_j^{-1} \to x_0$ locally uniformly in $\overline{\mathbb{R}}^n - \{x_0\}$ and $\overline{\mathbb{R}}^n - \{y_0\}$, respectively, as $j \to \infty$.

Convergence groups are similarly defined on other spaces. Quasiconformal groups are convergence groups. A convergence group is *discrete* if (1) above never holds. The elements of a discrete convergence group fall into three categories: elliptic, parabolic, and loxodromic. Examples are obtained from the following: If E is a totally disconnected closed set in $\overline{\mathbb{R}}^n$ and if a group G of self-homeomorphisms of $\overline{\mathbb{R}}^n$ is properly discontinuous in $\overline{\mathbb{R}}^n \setminus E$, then G is a discrete convergence group whose limit set lies in E.

These initial studies led to further attempts to generalize to higher dimensions the universal geometric constraints on Fuchsian groups given by Beardon. A first step entailed specifying the connection between criteria for discrete groups and holomorphic dynamics—the iteration of polynomials of one complex variable on the Riemann sphere (Gehring and Martin 1989). This advance was already inherent in Jørgensen's proof of his inequality for discrete groups. Meanwhile, at Mittag-Leffler in 1990 Gehring and Martin discovered entirely new classes of polynomial trace identities and thus a tool to study the geometry of Kleinian groups.

As a result, a number of new developments ensued. Among the more important papers was Gehring and Martin (1994). Jane Gilman wrote:

> *This paper is the seventh in a series of 10 remarkable papers that Gehring and Martin have written on the geometry of discrete Möbius groups .... The papers use a combination of hyperbolic geometry and complex iteration theory to obtain discreteness criteria for Möbius groups. The results obtained in the current paper include a sharp analogue (for subgroups of PSL(2, ℂ) with an elliptic element) to the Shimizu-Leutbecher inequality... and the elimination of a large region of possible values for the commutator parameter (trace[g, h] – 2 with g, h ∈ G) of a discrete group G with two elliptic generators. From this, the authors are able to obtain a sharp lower bound for the distance between the axes of elliptics of order n*

and m in any discrete groups (for many values of m and n). The results have as a corollary substantial improvements…on volumes of hyperbolic orbifolds….

Even with these new tools at hand, it took quite some time to claim the real prize, which came in the paper "Minimal Co-Volume Hyperbolic Lattices. I. The Spherical Points of a Kleinian Group" (Gehring and Martin 2009)—Gehring's last research article. This paper, along with its sequel (Marshall and Martin 2012) completed the identification of the hyperbolic 3-orbifold of smallest volume solving a problem of Siegel from 1942 and also Problem 3.60 (F) in the Kirby problem list.

Quasidisks

A *quasidisk* Ω is the image of the unit disk $\mathbb{D}$ under a quasiconformal mapping of the complex plane $\mathbb{C}; f:\mathbb{C}\to\mathbb{C}$ and $f(\mathbb{D})=\Omega$. Thus a quasidisk is a simply connected planar domain with reasonable geometric control on the structure of its boundary. The importance of the concept is reflected in the many remarkable and diverse applications in planar geometric function theory and in low-dimensional geometry and topology, where quasidisks form the components of the limit sets of quasifuchsian groups. In dynamics, quasidisks form the components of the filled-in Julia set of a hyperbolic rational map. Further, a holomophic perturbation of the unit disk in the space of injections gives a quasidisk. It was another lifelong task of Gehring's to collect characterizations of quasidisks, to give new and different proofs for them, to find new applications of the theory, and to generally spread the word about these wonderful objects. The only book that Gehring wrote in his lifetime was the monograph written with his former student Kari Hag, *The Ubiquitous Quasidisk* (Gehring and Hag 2012).

This book can best be thought of as a cross-section of the many connections between the theory of planar quasiconformal mappings, geometric function theory, and analysis. Ahlfors' criterion been known for rather a long time: The Jordan domain Ω is a quasidisk if and only if there exists a constant *c* such that, for all pairs of points,

$$z_1, z_2 \in \partial\Omega \ \min_{j=1,2}\left\{\operatorname{diam}(\gamma_j) \le dc\,|\,z_1 - z_2\,|\right\},$$

where γ_1, γ_2 are the components of $\partial\Omega\setminus\{z_1, z_2\}$. This is typical of the geometric characterizations that include the notions of linear local connectedness. Other geometric characterizations are based, for example, on the estimates on the hyperbolic or quasihyperbolic metric.

Function-theoretic characterizations include, for instance, the Schwarzian derivative property. For an analytic φ,

$$S\varphi = \left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2,$$

and Ω is a quasidisk if and only if there exists a constant $c > 0$ such that f is injective whenever f is analytic in Ω with $|S\varphi| \leq c p_{\Omega}^2, \varphi' \neq 0$ in Ω. Of course, the Schwarzian derivative is Möbius invariant and has wide application in modular forms, hypergeometric functions, and Teichmüller theory. Further criteria found by Gehring and Hag concerned the injectivity properties of quasi-isometries, extension properties of BMO, or $W^{1,2}$ spaces (and their near relatives). We recommend that the reader thumb through the Gehring and Hag book to see the vast and diverse collection of results, which cover a broad spectrum of planar function theory.

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