NATIONAL ACADEMY OF SCIENCES

RICHARD COURANT
1888–1972

A Biographical Memoir by

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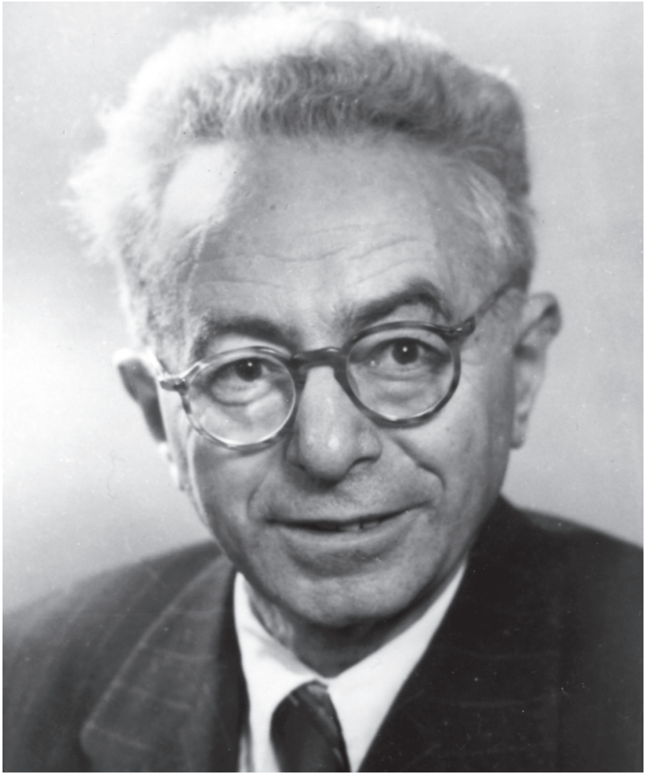
Biographical Memoirs, VOLUME 82

PUBLISHED 2003 BY
THE NATIONAL ACADEMY PRESS
WASHINGTON, D.C.



RICHARD COURANT

January 8, 1888–January 27, 1972

BY PETER D. LAX

DURING HIS LONG and adventurous life Courant achieved many things in mathematics: in research and the applications of research, in the exposition of mathematics and the education of students, and in administrative and organizational matters. To understand how he, essentially an outsider both in Germany and the United States, accomplished these things we have to examine his personality as well as his scientific works. But let's start at the beginning.

Courant was born on January 8, 1888, in the small town of Lublinitz in Upper Silesia, now part of Poland but then of Germany. His father, Siegmund, was an unsuccessful businessman. The family moved to Breslau, and soon the precocious Richard was beginning to support himself by tutoring. In the *gymnasium* he came under the influence of a charismatic teacher of mathematics, Maschke, who inspired specially selected, talented students with a love of mathematics. Six years after Courant the young Heinz Hopf entered the *gymnasium* in Breslau and came under the tutelage of Maschke, who trained his special pupils by posing challenging problems. Many years later Hopf recalled that he was able to solve most of them, but was stumped every once in a while. "Courant could solve it," said Maschke.

This no doubt was the first bond in the intimate friendship that developed later between Courant and Hopf.

After the *gymnasium* Courant was ready to attend university lectures on mathematics and physics at the University of Breslau. Because of the weakness of the physics faculty he gravitated toward mathematics. He spent a semester at the University of Zürich, but still dissatisfied, he set out in the fall of 1907 for what his Breslau friends, Otto Toeplitz and Ernst Hellinger, described as the Mecca of mathematics, Göttingen.

Not long after his arrival in Göttingen Courant was accepted as a member of the "in-group" of young mathematicians whose leader was Alfred Haar, and which included Toeplitz, Hellinger, and von Kármán but did not include Hermann Weyl. These brilliant young men were attracted to Göttingen by its stellar mathematics faculty: Klein, Hilbert, Minkowski, Runge, Zermelo, the fluid dynamicist Prandtl, and the astrophysicist Schwarzschild. For young Courant the shining light was Hilbert, and it was his great good fortune that in 1908 Hilbert chose him to be his assistant.

The next phase in his career was the writing of a dissertation. It is illuminating to go back more than 50 years to the dissertation of Riemann. Riemann proved the existence of harmonic functions by a variational method called Dirichlet's principle, according to which a certain quadratic functional—the Dirichlet integral—for functions of two variables assumes its minimum value. Weierstrass challenged the validity of this proof, because the existence of a minimum cannot be taken for granted. Weierstrass even gave an example of a fourth order functional whose minimum is not assumed by any function. In view of the basic nature of Riemann's work, there was a feverish effort by the leading mathematicians to furnish an alternative proof. Poincaré did it with a method he called "balayage." Carl Neumann derived and then solved an integral equation; this work paved the way to Fredholm's famous study of general integral equations, which in turn was followed by Hilbert's and subsequently Frederic Riesz's analysis of the spectrum of compact operators. Herman Amandus Schwarz supplied the missing proof by his famous "alternating method." One hundred years later, by one of those twists that are not infrequent in mathematics, the alternating method, now called "domain decomposition," turned out to be the most efficient *numerical* method for solving Riemann's problem and more general problems, when the calculations are performed by computers with many processors in parallel.

Hilbert's way of filling the gap was to supply the missing step in Riemann's argument, the existence of the minimizing function. Curiously, according to Haim Brezis and Felix Browder, Hilbert's own work is incomplete. The crucial idea for fixing Dirichlet's principle was supplied in 1906 by Beppo Levi.

Back to Courant. Hilbert suggested to him as a dissertation topic to use Dirichlet's principle to prove the existence of various classes of conformal maps. Courant succeeded, and was awarded his Ph.D. summa cum laude in 1910. The same topic served for his *habilitation* dissertation in 1912.

Dirichlet's principle remained a lodestar for Courant; he kept returning to it throughout his career. He was fascinated not only by its use in theory but also by the possibility of basing numerical calculations on it, as was done by the young physicist Walther Ritz.

Courant liked to spice his lectures with remarks about the personalities of scientists, to render them more human. Thus, in a talk in Kyoto in 1969, his last public lecture, he described the work of Walther Ritz and recalled that Ritz died young, at the age of 31, of tuberculosis, and that he refused to enter a sanitarium for fear that it would prevent him from completing his life's work. Then Courant added that Walther Ritz was a member of the Swiss family whose hotels all over the world made their name synonymous with luxury.

In 1912 Courant married Nelly Neumann, a fellow student from Breslau; the marriage lasted only four years. They were joined in Göttingen by Courant's favorite intellectual cousin from Breslau, Edith Stein, who became a student and later assistant to the philosopher Husserl. She attained martyrdom as a Jew and posthumous fame as a saint of the Roman Catholic Church, canonized by Pope John Paul II in 1998.

Harald Bohr turned up in Göttingen in 1912; he and Courant became fast friends. They wrote a joint paper on the distribution of the values of the Riemann ζ function along the lines Re ζ = const in the critical strip. This was Courant's only venture into number theory. The friendship with Harald Bohr later came to include Harald's brother, Niels, and lasted until the end of Niels's life.

The idyllic Göttingen life was shattered, like everything else in Europe, by the outbreak of the First World War; the flower of European youth was led to slaughter. Courant was drafted into the army; he fought on the western front and was seriously wounded. While in the trenches, Courant had seen the need for reliable means of communication, and came to the idea of a telegraph that would use Earth as a conductor. He consulted Telefunken, the German telephone company, and his teacher Carl Runge in Göttingen, who brought Peter Debye and Paul Scherrer into the project. In the end the Earth telegraph became a resounding success; equally important, the experience taught Courant how to deal successfully with people of all classes: officers of rank high and low, engineers, industrialists.

Courant's absence in the army did not make his and Nelly's hearts grow fonder. On the contrary, it made both of them realize their incompatibility. After their divorce in 1916 he found himself drawn to Nina Runge, daughter of Carl Runge, professor of applied mathematics in Göttingen, and she to him. They were married in 1919. They had much in common—a passionate love of music—but in many respects they were very different. Their marriage was a successfully shared life. They had four children: two boys, who became physicists, and two girls, a biologist and a musician.

The years 1918-20 were banner years for Courant. He proved that among all plane domains with prescribed perimeter, the circle had the lowest fundamental frequency. This was followed by a max-min principle that enabled him to determine the asymptotic distribution of eigenvalues of the Laplace operator over any domain, a result of great physical interest, established previously by Weyl with the aid of a min-max principle. Weyl's method leads naturally to upper bounds for the eigenvalues, Courant's to lower bounds. The combination of the two methods is particularly effective.

It was during this period that Courant's friendship with the publisher Ferdinand Springer matured. Courant encouraged Springer to enlarge his offering in mathematics. This led to Springer's taking over the *Mathematischen Annalen* and starting the *Mathematischen Zeitschrift*. Equally important was the new book series "Grundlehren," affectionately known as the Yellow Peril for its yellow cover.

After the war Courant was offered and accepted a professorship in Münster, but this was merely a steppingstone for a position the following year in Göttingen, pushed through by Hilbert and Klein. The latter saw Courant—correctly—

as one who would share his vision of the relation of mathematics to science, who would seek a balance between research and education, and who would have the administrative energy and savvy to push his mission to fruition.

The early 1920s were a tough time in Germany. The defeat in the First World War had demoralized large segments of society and had led to rampant inflation. Courant showed his resourcefulness by keeping things afloat, partly with the help of the far-sighted industrialist Carl Still.

In 1922 Courant's first book appeared, *Hurwitz-Courant on Function Theory*, the third volume in the Yellow Peril series. The first part, based on lecture notes of Hurwitz, was written from the Weierstrass point of view; its main subject was elliptic functions. Courant supplemented this material with nine chapters on Riemann surfaces, conformal mapping, and automorphic functions. Courant used an informal, intuitive notion of a surface that displeased some readers but pleased others.

Two years later, in 1924, the first volume of Courant-Hilbert appeared. It was based on lecture notes of Hilbert but even more on Courant's own research in the past five years. The book starts with a 40-page chapter on linear algebra, presented from an analytic point of view, so that generalization to infinite dimension comes naturally. This is followed by chapters on orthogonal function systems, the Fredholm theory of integral equations, the calculus of variations, and the vibrations of continuum mechanical systems, using extensively the spectral theory of self-adjoint ordinary and partial differential operators. In 1926 Schrödinger invented his wave mechanics, formulated in terms of partial differential operators and their eigenvalues and eigenfunctions. Fortuitously, Courant-Hilbert Volume I contained much of the mathematics needed to understand and solve

Schrödinger's equations. This was a striking example of mathematics anticipating the needs of a new physical theory.

In 1928 Courant, Friedrichs, and Lewy published their famous paper on the partial difference equations of mathematical physics. The main motivation for writing it was to use finite difference approximations to prove the existence of solutions of partial differential equations. The paper discusses elliptic, parabolic, and hyperbolic equations; it contains a wealth of ideas, such as the probabilistic interpretation of elliptic difference equations, and the restriction that has to be imposed on the ratio of the time increment and the space increment. The latter, known as the CFL condition, became famous during the computer age. Woe to the computational scientist who ignorantly violates it. This is an outstanding example of research undertaken for purely theoretical purposes turning out to be of immense practical importance.

In 1927 Volume 1 of Courant's calculus text appeared, soon followed by Volume 2. It has been extremely successful in every sense; its translation into English by McShane has sold 50,000 copies of Volume 1 and 35,500 copies of Volume 2 in the United States. It has shaped the minds of many who wanted and needed a deeper grasp of the calculus. Even after 70 years it is better than most, nay all, calculus books in use today in the United States.

In the 1920s and early 1930s Göttingen became again a Mecca of mathematics, as well as of physics. A list of visitors, long term or short term, reads like a Who's Who of mathematicians: Alexandrov, Artin, Birkhoff, Bohr, Hopf, Hardy, Khinchin, Kolmogorov, Lyusternik and Shnirelman, MacLane, von Neumann, Nielsen, Siegel, Weil, Weyl, Wiener, and many others. Paul Alexandrov described the atmosphere thus: "[Göttingen was] one of the principal centers of world mathematical thought; the place to which all mathematicians came from all over the world, of all possible trends and ages, where there was an exchange of all mathematical ideas and discoveries as soon as they had arisen, no matter where. . . .”

There were many assistants and postdocs around; Courant had private sources of money to pay their stipends. This caused some confusion after the Second World War, when the German government, to its credit, decided to compensate not only faculty members who were dismissed by the Nazis but assistants as well. Many of the assistants in Göttingen thus dismissed had a hard time establishing their claim, for their names did not appear on the roster of those whose salary was paid by the university.

In 1926 concrete negotiations were started, and plans laid, for housing the institutes of mathematics and physics in a permanent building, for long a dream of Felix Klein, now enthusiastically taken up by Courant. The International Educational Board of the Rockefeller Foundation agreed to supply $350,000, and the Prussian Ministry of Education agreed to cover the maintenance costs.

The building of the institute was finished and dedicated in 1929. Courant became its director. Yet this moment of triumph already contained the seed of its own destruction, and that of most civilized Western institutions. The stock market in the United States crashed a few months earlier, leading to a deep economic depression that soon became worldwide. The misery caused by this drove a sizeable part of the German voting population, already embittered by the defeat in the First World War, to support the Nazis. In January 1933 the Hitler gang took over the government. It soon established a new age of barbarism in Germany. For a start, Jewish employees of the state, including professors, were dismissed summarily, Courant among the first. For once his grasp of reality deserted him, and he went from pillar to post to be reinstated. A chance encounter with a member of the Nazi party, who was a member of the university community, set him right. “No doubt you and your friends believe that the excesses of the first few months of the new regime will die down and everything will return to how it was before,” said the man. “You are mistaken. Things will get worse and worse for you. You had better get out while you can.”

Get out Courant did. After a brief stay in England he, his family, and some family friends landed in New York, where thanks to Oswald Veblen and Abraham Flexner, a position was offered to him in the department of mathematics of New York University, with the charge to develop a graduate program. His host there was the mathematician Donald Flanders, admired by all for his saintly character. He and Courant formed a deep friendship that today extends to their children.

At NYU Courant found a mathematical desert. How he made it bloom is a fascinating story. It started in 1936 with a burst of creative energy. He showed how to solve Plateau’s problem—finding a minimal surface spanning a given contour in space—by using Dirichlet’s principle. A solution of this classical problem had been found earlier by Jesse Douglas, and in another way by Tibor Radó, but the elegance and simplicity of Courant’s method had opened the way to attack more general problems concerning minimal surfaces, such as minimal surfaces spanning multiple contours, of higher genus, and having part of their boundary restricted to a prescribed surface. Courant pursued these generalizations during the next 10 years. The culmination was the book *Dirichlet’s Principle, Conformal Mapping, and Minimal Surfaces* that appeared in 1950.

In 1937 Courant was joined at NYU by his brilliant former student from Göttingen, K. O. Friedrichs, and by James J.

Stoker, an American. Stoker's original training had been in engineering. In the 1930s, mid-career, he decided to seek a Ph.D. in mechanics at the Federal Institute of Technology in Zürich. One of the first courses he took there was by Heinz Hopf on geometry. Stoker was so charmed by the subject, and the teacher, that he switched his doctoral studies to differential geometry. Hopf wrote to Courant to call his attention to this *junger Amerikaner* whose scientific outlook and temperament were so close to Courant's.

With Friedrich's help Courant was able to complete the long-awaited second volume of Courant-Hilbert. It was, along with Hadamard's book on the Cauchy problem, the first modern text on partial differential equations.

In 1941 *What Is Mathematics?* by Courant and Robbins appeared, a highly popular book written "for beginners and scholars, for students and teachers, for philosophers and engineers, for classrooms and libraries." In the preface Courant warns about the danger facing the traditional place of mathematics in education, and outlines what to do about it. It ought to be compulsory reading for all who today are engaged in reforming the teaching of mathematics.

Courant found in New York City a "vast reservoir" of talented young people, and he was eager to attract them to study mathematics at NYU. To enable those who worked during the day to attend classes, graduate courses were offered in the evening, once a week, for two hours at a time.

America's entry into the Second World War transformed most American academic scientific institutions, none more than Courant's operation at NYU. Government funding was made available for research relevant to war work through the Office of Scientific Research and Development (OSRD). Its head, Vannevar Bush, saw the importance of mathematics for the war effort and set up the Applied Mathematics

Panel under the direction of Warren Weaver. Courant was soon invited to be a member of this elite group.

The mathematical project at NYU sponsored by the OSRD was about the flow of compressible fluids in general and the formation and propagation of shock waves in particular. There was enough money to support young research associates (Max Shiffman, Bernard Friedman, and Rudolf Lüneburg), who also served as adjunct faculty in the graduate school. There was also money to provide stipends for graduate students, some of whom were drawn into war work. Courant insisted that graduate training continue even during the war.

This is a good place to describe Courant as a classroom teacher. He seldom bothered to prepare the technical details of his lecture. He muttered in a low voice, and his writing was often indecipherable. Nevertheless, he managed to convey the essence of the subject and left the better students with a warm glow of belief that they could nail down the details better than the master.

Courant supervised many graduate students' doctoral dissertations, more than 20 in Göttingen and a like number at NYU. Among the former were Kurt Friedrichs, Edgar Krahn, Reinhold Baer, Hans Lewy, Otto Neugebauer, Willi Feller, Franz Rellich, Rudolf Lüneburg, Herbert Busemann, and Leifur Asgeirson. In the United States he taught Max Shiffman, Joe Keller, Harold Grad, Avron Douglis, Martin Kruskal, Anneli Lax, Herbert Kranzer, and Donald Ludwig—a very fine record.

The bulk of the research conducted during the war was fashioned into a book on supersonic flows and shock waves. The editing was in the hands of Cathleen Morawetz, who had the delicate task of reconciling Courant's freely flowing exposition with Friedrichs's demand for precision. *Supersonic Flow and Shock Waves* appeared in book form in

1948; it was a very useful and successful treatise, with a mathematical flavor, on the flow of compressible fluids.

As the war neared its end Courant's thoughts turned to postwar developments. He shrewdly realized that after the war the U.S. government would continue to support science. The critical contribution of science to the war effort had been noted by statesmen: radar, the proximity fuse, bombsights, code breaking, aerodynamic design, and the atomic bomb. Courant also realized that applied mathematics would be an important part of the government's plans, and he successfully used his wartime contacts to gain support for his vision of applied mathematics. Support came from the office of Naval Research, the Atomic Energy Commission, the offices of Army and Air Force research, and later from the National Science Foundation. As always, he emphasized that research must be combined with teaching.

When it came to hiring faculty, Courant relied on his intuition. The candidate's personality was often more important than the field he was working in. Courant did not like following fashions and fads. "I am against panic buying in an inflated market" was his motto.

Courant did not extend his hatred of Nazis to the German nation. He overcame his resentments and was eager to help those deserving help. He traveled to Germany as soon as it was possible, in 1947, to see the situation there firsthand, to talk to people he trusted. He arranged visits to the United States for a number of young mathematicians; this had a tremendous effect psychologically and scientifically and earned Courant the gratitude of the younger generation of German mathematicians and later the Knight Commander's Cross of the Order of Merit of the Federal Republic of Germany.

In 1948 Courant's sixtieth birthday was celebrated with much emotion by mathematicians invited from both sides of the Atlantic; nostalgia flowed like water, held within bounds by Courant's natural irony.

In 1954 the Atomic Energy Commission decided to place one of its supercomputers, the UNIVAC, at a university. After a fierce competition Courant's institute was chosen. The UNIVAC had a memory of 1,000 words and used punched cards. The commission repeatedly replaced it with newer models; the last one was a CDC 6600, installed in 1966 and put out to pasture in 1972.

Courant retired in 1958 at age 70; his successor was Stoker. In retirement Courant succeeded in finishing the translation into English of Volume 2 of *Methods of Mathematical Physics*. This was no mere translation. Courant made a serious attempt, with much help from younger colleagues, to update the material from a mere 470 to over 800 pages. The book ends with a 30-page essay written by Courant on ideal functions, such as distributions. The last chapter in the German original, on existence proofs using variational methods, was omitted. Courant planned to rework it, together with a discussion of finite difference methods, and to issue it as Volume 3. Alas, he was not up to the task.

Courant was deeply concerned about the Cold War. He felt that the natural comradeship of scientists, in particular of mathematicians, might set an example and overcome the "us versus them" stereotypes. Accordingly, he was among the first to visit the Soviet Union. The time—the summer of 1960—was not auspicious, for the Soviets had just shot down a U.S. U-2 spy plane. The remains of the plane and the spy paraphernalia were displayed in the middle of Moscow's Gorky Park. There was a long line of curiosity seekers. As a distinguished visitor, Courant was whisked to the head of the line and was introduced to the aeronautical engineer who was there to explain the workings of the U-2. The engineer was deeply honored: "Professor Courant, I learned aerodynamics from your book." It had been translated in 1950 into Russian, as were all of Courant's other books.

In 1963 Courant led a delegation of about 15 U.S. mathematicians to a two-week conference on the occasion of the opening of the Academic City and University at Novosibirsk. It was a golden time and gave rise to friendships that lasted lifetimes.

A very generous gift from the Sloan Foundation, augmented by the Ford Foundation and the National Science Foundation, was used to construct a handsome 13-story building just off Washington Square, in which the Courant Institute, so named at its dedication in 1965, still nestles. Its architects won all kinds of prizes and went on to build for departments of mathematics at Princeton and Rutgers.

Courant's last years were full of recognition and honors. Solomon Lefschetz admired Courant for having built an enduring school of mathematics, and had nominated him to receive a National Medal of Science. None of these encomiums, however, could lift Courant's spirits in his extreme old age. His institute was thriving, his children and grandchildren were happily launched on careers, but none of that would dissipate his gloom. His old stratagems to overcome depression—embark on a new project, make the acquaintance of a fascinating woman—were no longer available to him. He even stopped playing the piano, which had been a great source of pleasure for him in the past, a way of transcending conflicts and disappointments. He died on January 27, 1972, at the age of 84.

It is time to look back and ask what manner of man was Courant. For this we must look at the testimony of people who knew him intimately. Surprisingly, they were utterly different from Courant in many, sometimes all ways, such as Flanders, a descendent of Puritans and a puritan himself. Flanders was haunted by a lack of confidence. He loved and needed Courant's ebullience, and Flanders's wit and pure spirit were deeply necessary to Courant.

There was Otto Neugebauer, a meticulous and workaholic scholar, about whom Courant said that "he had all the virtues and none of the faults of pedantry." Neugebauer in turn described Courant's style of operation thus:

All that lies before us as scientific achievement and organization seems to be the outcome of a well conceived plan. To us, nearby, things seemed sometimes more chaotic than planned, and we were far from always in agreement. But a never failing loyalty bound Courant's associates together; he inspired an unshakable confidence in his profound desire to do what was right and what made sense under the given circumstances. His ability to create a feeling of mutual confidence in those who know him intimately lies at the foundation of his success and influence.

Friedrichs was another former student utterly devoted to Courant, although totally different from him. He described the excitement he felt when as a young student he read Courant's presentation of geometric function theory in Hurwitz-Courant.

It is true that there were some passages in which matters of rigor were taken somewhat lightly, but the essence came through marvelously. I was reminded of this effect much later, when I heard Courant play some Beethoven piano sonata. There were also some difficult passages which he somehow simplified; but the essence carried over wonderfully. In a way, one could perhaps say the same thing about his skiing—a sport which, incidentally, his assistants were expected to have mastered, or else to learn from him. Never mind the details of the operations, he always managed to come down the mountain quite safely.

Here are the observations of another close friend, Lucile Gardner Wolff.

There are some great men—and among them some of the greatest—who owe their preeminence not merely to their good qualities but to their bad, or to what would have been bad in another man; whose talents, however remarkable, cannot in themselves account for their achievements or explain why they succeeded where others, equally gifted, failed; whose genius lay precisely in the ability to turn their weak points to good account. Such a one is the late Richard Courant.

Again Friedrichs:

As a person Richard Courant cannot be measured by any common standard. Think of it: a mathematician who hated logic, who abhorred abstractions, who was suspicious of truth—if it was just bare truth. For a mathematician these seem to be contradictions. But Richard Courant was never afraid of contradictions—if they could enhance the fullness of life.

Courant loved to be with young people. He understood their ambitions and anxieties and was ready with useful, often unconventional advice. For many the Courant Institute was a second family, and some of this spirit abides today.

Courant has been gone for nearly 30 years, surely enough time for the verdict of history. What is remembered? His insistence on the fundamental unity of all mathematical disciplines and on the vital connection between mathematics and other sciences. The name he gave to his institute—Institute of Mathematical Sciences—expressed his attitude. It is remarkable that today many of the leading mathematical institutions have adopted this appellation.

Courant insisted that research be combined with teaching, a philosophy he liked to trace back to the French Revolution and the founding of the Ecole Polytechnique.

Courant was a superb writer, in both German and English. Three of his books—*What Is Mathematics?*, his two volumes of calculus, and *Supersonic Flow and Shock Waves*, written with Friedrichs—are alive and well. Even his occasional pieces are worth re-reading. For example, in an article on mathematics in the modern world, which appeared in 1964 in *Scientific American,* he wrote,

To handle the translation of reality into the abstract models of mathematics and to appraise the degree of accuracy thereby attainable calls for intuitive feelings sharpened by experience. It may also often involve the framing of genuine mathematical problems that are far too difficult to be solved by the available techniques of the science. Such, in part, is the nature of the intellectual adventure and the satisfaction experienced by the mathematician who works with engineers and natural scientists on the mastering of the real problems that arise in so many places as man extends his understanding and control of nature.

Those who wish to find out more about Courant's adventurous life can learn much from Constance Reid's biography subtitled "The Story of an Improbable Mathematician." Those who knew him remember his great warmth and kindness, often disguised by irony, his energy and enthusiasm coupled with skepticism, and his inexhaustible optimism in the face of seemingly insurmountable obstacles.

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